

Inequalities among eigenvalues of different self-adjoint discrete Sturm-Liouville problems

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Abstract. In this paper, inequalities among eigenvalues of different self-adjoint discrete Sturm-Liouville problems are established. For a fixed discrete Sturm-Liouville equation, inequalities among eigenvalues for different boundary conditions are given. For a fixed boundary condition, inequalities among eigenvalues for different equations are given. These results are obtained by applying continuity and discontinuity of the n -th eigenvalue function, monotonicity in some direction of the n -th eigenvalue function, which were given in our previous papers, and natural loops in the space of boundary conditions. Some results generalize the relevant existing results about inequalities among eigenvalues of different Sturm-Liouville problems.

Keywords: inequality; discrete Sturm-Liouville problem; eigenvalue; self-adjointness.

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1. Introduction

A self-adjoint discrete Sturm-Liouville problem (briefly, SLP) considered in the present paper consists of a symmetric discrete Sturm-Liouville equation (briefly, SLE)

$$-\nabla(f_n \Delta y_n) + q_n y_n = \lambda w_n y_n, \quad n \in [1, N], \quad (1.1)$$

and a self-adjoint boundary condition (briefly, BC)

$$A \begin{pmatrix} y_0 \\ f_0 \Delta y_0 \end{pmatrix} + B \begin{pmatrix} y_N \\ f_N \Delta y_N \end{pmatrix} = 0, \quad (1.2)$$

where $N \geq 2$ is an integer, Δ and ∇ are the forward and backward difference operators, respectively, i.e., $\Delta y_n = y_{n+1} - y_n$ and $\nabla y_n = y_n - y_{n-1}$; $f = \{f_n\}_{n=0}^N$, $q = \{q_n\}_{n=1}^N$ and $w = \{w_n\}_{n=1}^N$ are real-valued sequences such that

$$f_n \neq 0 \text{ for } n \in [0, N], \quad w_n > 0 \text{ for } n \in [1, N]; \quad (1.3)$$

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λ is the spectral parameter; the interval $[M, N]$ denotes the set of integers $\{M, M + 1, \dots, N\}$; A and B are 2×2 complex matrices such that $\text{rank}(A, B)=2$, and satisfy the following self-adjoint boundary condition:

$$A \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} A^* = B \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} B^*, \quad (1.4)$$

where A^* denotes the complex conjugate transpose of A .

Throughout this paper, by \mathbb{C} , \mathbb{R} , and \mathbb{Z} denote the sets of the complex numbers, real numbers, and integer numbers, respectively; and by \bar{z} denote the complex conjugate of $z \in \mathbb{C}$. Moreover, when a capital Latin letter stands for a matrix, the entries of the matrix are denoted by the corresponding lower case letter with two indices. For example, the entries of a matrix C are c_{ij} 's.

As it is mentioned in [1, 7], the discrete SLP (1.1)–(1.2) can be applied to many fields, ranging from mechanics, to network theory, and to probability theory. The eigenvalues of (1.1)–(1.2) play an important role in studying these physical problems and they change as the SLP changes. Thus, it is naturally important to compare the eigenvalues of different SLPs. In this paper, we shall establish inequalities among eigenvalues of different SLPs.

Recall that a self-adjoint continuous SLP consists of a differential SLE

$$-(p(t)y')' + q(t)y = \lambda w(t)y, \quad t \in (a, b), \quad (1.5)$$

and a BC

$$A \begin{pmatrix} y(a) \\ (py')(a) \end{pmatrix} + B \begin{pmatrix} y(b) \\ (py')(b) \end{pmatrix} = 0, \quad (1.6)$$

where $-\infty < a < b < +\infty$; $1/p, q, w \in L((a, b), \mathbb{R})$, $p, w > 0$ almost everywhere on (a, b) , while $L((a, b), \mathbb{R})$ denotes the space of Lebesgue integrable real functions on (a, b) ; A and B are 2×2 complex matrices such that $\text{rank}(A, B)=2$ and (1.4) holds. For a fixed differential SLE (1.5), inequalities among eigenvalues for different self-adjoint BCs have been extensively studied by many authors (cf., e.g. [3–6, 8, 10, 12–15, 20, 21]). Using the variational method, Courant and Hilbert in [5] gave inequalities among eigenvalues for different separated BCs. Using the Prüfer transformation of (1.5), Coddington and Levinson in [4] gave the classical inequalities among eigenvalues for periodic, antiperiodic, Dirichlet and Neumann BCs under some conditions on the coefficients of (1.5). See also [20]. For an arbitrary coupled self-adjoint BC, Eastham and his coauthors in [6, Theorem 3.2] identified two separated BCs corresponding to the Dirichlet and Neumann BCs in the above case, and established analogous inequalities. Their proof also depends on the Prüfer transformation of (1.5). See also [8]. These inequalities are extended to singular SLPs and other cases [3, 10, 12–14]. Using natural loops in the space of self-adjoint BCs,

Peng and his coauthors in [15] gave a short proof of [6, Theorem 3.2], and obtained new general inequalities. See Theorem 4.53 in [15].

Next, we recall the related existing results of inequalities among eigenvalues of different self-adjoint discrete SLPs (1.1)–(1.2). For a fixed self-adjoint BC (1.2), inequalities among eigenvalues for different coefficients of (1.1) were obtained by Rayleighs principles and minimax theorems in [16]. Then these results were extended to higher-order discrete vector SLPs in [17]. For a fixed equation (1.1), by using some oscillation results obtained in [1] and some spectral results of (1.1)–(1.2) obtained in [16], inequalities among eigenvalues for periodic, antiperiodic, and Dirichlet BCs were given under the assumption that $f_n > 0$, $1 \leq n \leq N - 1$, and $f_0 = f_N = 1$ in [19]. Under the same conditions of the coefficients of (1.1) and by a similar method used in [19], the above results in [19] were extended to a class of coupled BCs in [18].

The aim of the present paper is to establish more general inequalities among eigenvalues of different SLPs (1.1)–(1.2). For a fixed equation, inequalities among eigenvalues for different separated BCs are established in Theorem 3.1, and among eigenvalues for different BCs in a natural loop are established in Theorems 3.2–3.5. Then, the inequalities in Theorems 3.2–3.5 are applied to compare eigenvalues for coupled BCs with those for some certain separated ones (see Theorems 3.6–3.11), and eigenvalues for different coupled BCs (see Theorem 3.12). The inequalities in Theorems 3.6–3.11 extend those in [18, Theorem 3.1] to a more general case. For a fixed BC, inequalities among eigenvalues for equations with different coefficients and weight functions are established in Theorem 4.1, which generalize those in [16, Theorem 5.5] and [17, Theorem 3.6] in the second-order case. Combining the above results, one can establish inequalities among eigenvalues of SLPs with different equations and BCs (see Corollary 4.1 and Remark 4.2).

The method used in the present paper is different from those used in [4, 6, 8, 18–20]. On the one hand, the approaches used in [4, 6, 8, 20] in the continuous case depend on the Prüfer transformation of (1.5). Although the Prüfer transformation in discrete version were given in [2], some of its properties in continuous version can not be extended to the discrete one and thus similar methods used in [4, 6, 8, 20] are difficultly employed in studying the discrete problem. On the other hand, the variational method used in [16, 17] is restricted to compare eigenvalues, which have the same index, of different SLPs, and it seems to us that the method used in [18, 19] in the discrete case is hardly extended to a more general case. Note that the method used in [15] does not depend on the Prüfer transformation of (1.5). So the similarity of the self-adjoint BCs for continuous and discrete SLPs [11, 23] makes it possible and convenient to generalize a similar approach used in [15] to the discrete case. There are three major ingredients, which will be used by

this method: (1) the continuity and discontinuity of the n -th eigenvalue function, which were studied in [22]; (2) the monotonicity of the n -th eigenvalue function, which can be deduced from [22, 23]; (3) natural loops in the space of self-adjoint BCs, which can be obtained in a similar way as that in [15]. Thus, the work in the present paper, to some extent, can be regarded as a discrete analog of that in [15] and a continuation of our present works [22, 23].

This paper is organized as follows. Section 2 gives some preliminaries. Some notations are introduced and some lemmas are recalled. Especially, natural loops in space of self-adjoint BCs, are presented. In Section 3, inequalities among eigenvalues for different boundary conditions are given. In Section 4, inequalities among eigenvalues for different equations are established.

2. Preliminaries

In this section, some notations and lemmas are introduced. This section is divided into two parts. In Section 2.1, topology on space of SLPs and several useful properties of eigenvalues are recalled. In Section 2.2, natural loops in space of self-adjoint BCs established in [15] are presented.

2.1. Space of SLPs and properties of eigenvalues

Let the SLE (1.1) be abbreviated as $(1/f, q, w)$. Then the space of the SLEs can be written as

$$\Omega_N^{\mathbb{R},+} := \{(1/f, q, w) \in \mathbb{R}^{3N+1} : (1.3) \text{ holds}\},$$

and is equipped with the topology deduced from the real space \mathbb{R}^{3N+1} . Note that $\Omega_N^{\mathbb{R},+}$ has 2^{N+1} connected components. Bold faced lower case Greek letters, such as $\boldsymbol{\omega}$, are used to denote elements of $\Omega_N^{\mathbb{R},+}$.

The quotient space

$$\mathcal{A}^{\mathbb{C}} := M_{2,4}^*(\mathbb{C})/GL(2, \mathbb{C}),$$

equipped with the quotient topology, is taken as the space of general BCs; that is, each BC is an equivalence class of coefficient matrices of system (1.2), where

$$M_{2,4}^*(\mathbb{C}) := \{2 \times 4 \text{ complex matrix } (A, B) : \text{rank}(A, B) = 2\},$$

$$GL(2, \mathbb{C}) := \{2 \times 2 \text{ complex matrix } T : \det T \neq 0\}.$$

The BC represented by system (1.2) is denoted by $[A \mid B]$. Bold faced capital Latin letters, such as \mathbf{A} , are also used for BCs. The space of self-adjoint BCs is denoted by $\mathcal{B}^{\mathbb{C}}$. The following result gives the topology and geometric structure of $\mathcal{B}^{\mathbb{C}}$.

Lemma 2.1 [23, Theorem 2.2]. *The space $\mathcal{B}^{\mathbb{C}}$ equals the union of the following relative open sets:*

$$\begin{aligned}
\mathcal{O}_{1,3}^{\mathbb{C}} &= \left\{ \begin{bmatrix} 1 & a_{12} & 0 & \bar{z} \\ 0 & z & -1 & b_{22} \end{bmatrix} : a_{12}, b_{22} \in \mathbb{R}, z \in \mathbb{C} \right\}, \\
\mathcal{O}_{1,4}^{\mathbb{C}} &= \left\{ \begin{bmatrix} 1 & a_{12} & \bar{z} & 0 \\ 0 & z & b_{21} & 1 \end{bmatrix} : a_{12}, b_{21} \in \mathbb{R}, z \in \mathbb{C} \right\}, \\
\mathcal{O}_{2,3}^{\mathbb{C}} &= \left\{ \begin{bmatrix} a_{11} & -1 & 0 & \bar{z} \\ z & 0 & -1 & b_{22} \end{bmatrix} : a_{11}, b_{22} \in \mathbb{R}, z \in \mathbb{C} \right\}, \\
\mathcal{O}_{2,4}^{\mathbb{C}} &= \left\{ \begin{bmatrix} a_{11} & -1 & \bar{z} & 0 \\ z & 0 & b_{21} & 1 \end{bmatrix} : a_{11}, b_{21} \in \mathbb{R}, z \in \mathbb{C} \right\}.
\end{aligned} \tag{2.1}$$

Moreover, $\mathcal{B}^{\mathbb{C}}$ is a connected and compact real-analytic manifold of dimension 4.

Lemma 2.1 says that $\mathcal{O}_{1,3}^{\mathbb{C}}$, $\mathcal{O}_{1,4}^{\mathbb{C}}$, $\mathcal{O}_{2,3}^{\mathbb{C}}$, and $\mathcal{O}_{2,4}^{\mathbb{C}}$ together form an atlas of local coordinate systems on $\mathcal{B}^{\mathbb{C}}$.

The space $\Omega_N^{\mathbb{R},+} \times \mathcal{B}^{\mathbb{C}}$ of the SLPs is a real-analytic manifold of dimension $3N + 5$ and has 2^{N+1} connected components.

The following result gives the canonical forms of separated and coupled self-adjoint BCs.

Lemma 2.2 [21, Theorem 10.4.3]. *A separated self-adjoint BC can be written as*

$$\mathbf{S}_{\alpha,\beta} := \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 & 0 \\ 0 & 0 & \cos \beta & -\sin \beta \end{bmatrix}, \tag{2.2}$$

where $\alpha \in [0, \pi)$, $\beta \in (0, \pi]$; and a coupled self-adjoint BC can be written as

$$[e^{i\gamma} K \mid -I],$$

where $\gamma \in (-\pi, \pi]$, $K \in SL(2, \mathbb{R}) := \{2 \times 2 \text{ real matrix } M : \det M = 1\}$.

In particular, $\mathbf{S}_{0,\pi}$ is called the Dirichlet BC; $\mathbf{S}_{0,\beta}$ for any $\beta \in (0, \pi]$ or $\mathbf{S}_{\alpha,\pi}$ for any $\alpha \in [0, \pi)$ is called the BC which is Dirichlet at an endpoint. By \mathcal{B}_S and \mathcal{B}_C denote the space of separated self-adjoint BCs and that of coupled self-adjoint BCs, respectively. Then $\mathcal{B}^{\mathbb{C}} = \mathcal{B}_S \cup \mathcal{B}_C$, and \mathcal{B}_C is an open set of $\mathcal{B}^{\mathbb{C}}$.

Next, several properties of eigenvalues are presented. For each $\lambda \in \mathbb{C}$, let $\phi(\lambda) = \{\phi_n(\lambda)\}_{n=0}^N$ and $\psi(\lambda) = \{\psi_n(\lambda)\}_{n=0}^N$ be the solutions of (1.1) satisfying the following initial conditions:

$$\phi_0(\lambda) = 1, f_0 \Delta \phi_0(\lambda) = 0; \quad \psi_0(\lambda) = 0, f_0 \Delta \psi_0(\lambda) = 1.$$

Then the leading terms of $\phi_N(\lambda)$, $\psi_N(\lambda)$, $f_N \Delta \phi_N(\lambda)$, and $f_N \Delta \psi_N(\lambda)$ as polynomials of λ are

$$\begin{aligned}
&(-1)^{N-1} \left(\prod_{i=1}^{N-1} (w_i/f_i) \right) \lambda^{N-1}, \quad (-1)^{N-1} \left((1/f_0) \prod_{i=1}^{N-1} (w_i/f_i) \right) \lambda^{N-1}, \\
&(-1)^N \left(w_N \prod_{i=1}^{N-1} (w_i/f_i) \right) \lambda^N, \quad (-1)^N \left((w_N/f_0) \prod_{i=1}^{N-1} (w_i/f_i) \right) \lambda^N,
\end{aligned} \tag{2.3}$$

respectively. See [23] for details.

The following result says that the eigenvalues of a given SLP can be determined by a polynomial.

Lemma 2.3 [23, Lemma 3.2 and Lemma 3.3]. *A number $\lambda \in \mathbb{C}$ is an eigenvalue of each given SLP (1.1)–(1.2) if and only if λ is a zero of the polynomial*

$$\Gamma(\lambda) = \det A + \det B + G(\lambda),$$

where

$$G(\lambda) := c_{11}\phi_N(\lambda) + c_{12}\psi_N(\lambda) + c_{21}f_N\Delta\phi_N(\lambda) + c_{22}f_N\Delta\psi_N(\lambda),$$

$$c_{11} := a_{22}b_{11} - a_{12}b_{21}, \quad c_{12} := a_{11}b_{21} - a_{21}b_{11},$$

$$c_{21} := a_{22}b_{12} - a_{12}b_{22}, \quad c_{22} := a_{11}b_{22} - a_{21}b_{12}.$$

Let $(\omega, \mathbf{A}) \in \Omega_N^{\mathbb{R},+} \times \mathcal{B}^{\mathbb{C}}$. Set

$$r = r(\omega, \mathbf{A}) := \text{rank} \begin{pmatrix} -a_{11} + f_0 a_{12} & b_{12} \\ -a_{21} + f_0 a_{22} & b_{22} \end{pmatrix}. \quad (2.4)$$

Obviously, $0 \leq r \leq 2$. The following result establishes the relationship between analytic and geometric multiplicities of each eigenvalue of a given SLP and gives a formula for counting the number of eigenvalues.

Lemma 2.4 [23, Lemma 3.4 and Theorem 3.3]. *For any fixed self-adjoint SLP (1.1)–(1.2), all its eigenvalues are real, the number of its eigenvalues is equal to $N - 2 + r$, where r is defined by (2.4), and the analytic and geometric multiplicities of each of its eigenvalue are the same.*

Lemma 2.4 can also be deduced from [16, Theorem 4.1] and [17, Theorem 4.3]. By Lemma 2.4, we shall only say the multiplicity of an eigenvalue without specifying its analytic and geometric multiplicities. Based on these results, the problem (1.1)–(1.2) has $k = N - 2 + r$ eigenvalues (counting multiplicities), which can be arranged in the following non-decreasing order:

$$\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{k-1}.$$

The n -th eigenvalue λ_n can be considered as a function in the space of the SLPs, called the n -th eigenvalue function. The following result gives a necessary and sufficient condition for all the eigenvalue functions to be continuous in a set of space of SLPs.

Lemma 2.5 [22, Theorem 2.1]. *Let \mathcal{O} be a set of $\Omega_N^{\mathbb{R},+} \times \mathcal{B}^{\mathbb{C}}$. Then the number of eigenvalues of each $(\omega, \mathbf{A}) \in \mathcal{O}$ is equal if and only if all the eigenvalue functions restricted in \mathcal{O} are continuous. Furthermore, if \mathcal{O} is a connected set of $\Omega_N^{\mathbb{R},+} \times \mathcal{B}^{\mathbb{C}}$, then each eigenvalue function is locally a continuous eigenvalue branch in \mathcal{O} .*

2.2. Natural loops in the space of self-adjoint boundary conditions

In this subsection, natural loops in the space of self-adjoint BCs are presented. We remark that these natural loops will play an important role in studying inequalities among eigenvalues for coupled BCs and those for some certain separated ones.

Lemma 2.6 [15, Lemma 3.1]. *In $\mathcal{B}^{\mathbb{C}}$, we have the following limits:*

$$\begin{aligned}\mathbf{S}_1 &:= \lim_{s \rightarrow \pm\infty} \begin{bmatrix} 1 & s & \bar{z} & 0 \\ 0 & z & b_{21} & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & b_{21} & 1 \end{bmatrix}, \\ \mathbf{S}_2 &:= \lim_{t \rightarrow \pm\infty} \begin{bmatrix} 1 & a_{12} & \bar{z} & 0 \\ 0 & z & t & 1 \end{bmatrix} = \begin{bmatrix} 1 & a_{12} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \\ \mathbf{S}_3 &:= \lim_{s \rightarrow \pm\infty} \begin{bmatrix} s & -1 & \bar{z} & 0 \\ z & 0 & b_{21} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & b_{21} & 1 \end{bmatrix}, \\ \mathbf{S}_4 &:= \lim_{t \rightarrow \pm\infty} \begin{bmatrix} a_{11} & -1 & \bar{z} & 0 \\ z & 0 & t & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \\ \mathbf{S}_5 &:= \lim_{s \rightarrow \pm\infty} \begin{bmatrix} s & -1 & 0 & \bar{z} \\ z & 0 & -1 & b_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & b_{22} \end{bmatrix}, \\ \mathbf{S}_6 &:= \lim_{t \rightarrow \pm\infty} \begin{bmatrix} a_{11} & -1 & 0 & \bar{z} \\ z & 0 & -1 & t \end{bmatrix} = \begin{bmatrix} a_{11} & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\ \mathbf{S}_7 &:= \lim_{s \rightarrow \pm\infty} \begin{bmatrix} 1 & s & 0 & \bar{z} \\ 0 & z & -1 & b_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & b_{22} \end{bmatrix}, \\ \mathbf{S}_8 &:= \lim_{t \rightarrow \pm\infty} \begin{bmatrix} 1 & a_{12} & 0 & \bar{z} \\ 0 & z & -1 & t \end{bmatrix} = \begin{bmatrix} 1 & a_{12} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.\end{aligned}$$

The following result gives some natural loops in $\mathcal{B}^{\mathbb{C}}$.

Lemma 2.7 [15, Lemma 3.7].

(i) *Every BC $\mathbf{A} \in \mathcal{O}_{1,4}^{\mathbb{C}}$ lies on the following two simple real-analytic loops in $\mathcal{B}^{\mathbb{C}}$:*

$$\begin{aligned}\mathcal{C}_{1,4,z,b_{21}} &= \left\{ \mathbf{A}(s) := \begin{bmatrix} 1 & s & \bar{z} & 0 \\ 0 & z & b_{21} & 1 \end{bmatrix}, s \in \mathbb{R} \right\} \cup \{\mathbf{S}_1\}, \\ \mathcal{C}_{1,4,z,a_{12}} &= \left\{ \hat{\mathbf{A}}(t) := \begin{bmatrix} 1 & a_{12} & \bar{z} & 0 \\ 0 & z & t & 1 \end{bmatrix}, t \in \mathbb{R} \right\} \cup \{\mathbf{S}_2\}.\end{aligned}$$

(ii) *Every BC $\mathbf{A} \in \mathcal{O}_{2,4}^{\mathbb{C}}$ lies on the following two simple real-analytic loops in $\mathcal{B}^{\mathbb{C}}$:*

$$\begin{aligned}\mathcal{C}_{2,4,z,b_{21}} &= \left\{ \mathbf{B}(s) := \begin{bmatrix} s & -1 & \bar{z} & 0 \\ z & 0 & b_{21} & 1 \end{bmatrix}, s \in \mathbb{R} \right\} \cup \{\mathbf{S}_3\}, \\ \mathcal{C}_{2,4,z,a_{11}} &= \left\{ \hat{\mathbf{B}}(t) := \begin{bmatrix} a_{11} & -1 & \bar{z} & 0 \\ z & 0 & t & 1 \end{bmatrix}, t \in \mathbb{R} \right\} \cup \{\mathbf{S}_4\}.\end{aligned}$$

(iii) *Every BC $\mathbf{A} \in \mathcal{O}_{2,3}^{\mathbb{C}}$ lies on the following two simple real-analytic loops in $\mathcal{B}^{\mathbb{C}}$:*

$$\begin{aligned}\mathcal{C}_{2,3,z,b_{22}} &= \left\{ \mathbf{C}(s) := \begin{bmatrix} s & -1 & 0 & \bar{z} \\ z & 0 & -1 & b_{22} \end{bmatrix}, s \in \mathbb{R} \right\} \cup \{\mathbf{S}_5\}, \\ \mathcal{C}_{2,3,z,a_{11}} &= \left\{ \hat{\mathbf{C}}(t) := \begin{bmatrix} a_{11} & -1 & 0 & \bar{z} \\ z & 0 & -1 & t \end{bmatrix}, t \in \mathbb{R} \right\} \cup \{\mathbf{S}_6\}.\end{aligned}$$

(iv) Every BC $\mathbf{A} \in \mathcal{O}_{1,3}^{\mathbb{C}}$ lies on the following two simple real-analytic loops in $\mathcal{B}^{\mathbb{C}}$:

$$\begin{aligned}\mathcal{C}_{1,3,z,b_{22}} &= \left\{ \mathbf{D}(t) := \begin{bmatrix} 1 & s & 0 & \bar{z} \\ 0 & z & -1 & b_{22} \end{bmatrix}, s \in \mathbb{R} \right\} \cup \{\mathbf{S}_7\}, \\ \mathcal{C}_{1,3,z,a_{12}} &= \left\{ \hat{\mathbf{D}}(t) := \begin{bmatrix} 1 & a_{12} & \bar{z} & 0 \\ 0 & z & -1 & t \end{bmatrix}, t \in \mathbb{R} \right\} \cup \{\mathbf{S}_8\}.\end{aligned}$$

Remark 2.1. $\mathcal{C}_{1,4,z,b_{21}} \setminus \{\mathbf{A}\}$ is connected for any fixed $\mathbf{A} \in \mathcal{C}_{1,4,z,b_{21}}$. Similar result holds for other natural loops in (i)–(iv) of Lemma 2.7. For each $1 \leq i \leq 8$, \mathbf{S}_i is called a limit boundary condition (briefly, LBC) in the corresponding natural loop. Note that all the LBCs are separated ones.

3. Inequalities among eigenvalues for different boundary conditions

In this section, for any fixed equation, inequalities among eigenvalues for different BCs are established. This section is divided into four parts. In Subsections 3.1–3.4, inequalities among eigenvalues for different separated BCs, among eigenvalues for different BCs in a natural loop, among eigenvalues for coupled BCs and those for some certain separated ones, and among eigenvalues for different coupled BCs are established, respectively.

3.1. Inequalities among eigenvalues for separated BCs

In this subsection, we shall first compare the eigenvalues for different separated BCs $\mathbf{S}_{\alpha,\beta}$ in two directions α and β , separately. Then we give an application to compare eigenvalues for an arbitrary separated BC with those for the BCs which are Dirichlet at an endpoint.

For convenience, denote $\lambda_n(\alpha, \beta) := \lambda_n(\mathbf{S}_{\alpha,\beta})$ for short, and $\xi := \arctan(-1/f_0) + \pi$ if $f_0 > 0$; $\xi := \arctan(-1/f_0)$ if $f_0 < 0$.

Theorem 3.1. *Fix a difference equation $\omega = (1/f, q, w)$. Then $(\omega, \mathbf{S}_{\alpha,\beta})$ has exactly N eigenvalues if $\alpha \neq \xi$ and $\beta \neq \pi$; exactly $N - 1$ eigenvalues if either $\alpha \neq \xi$ and $\beta = \pi$ or $\alpha = \xi$ and $\beta \neq \pi$; and exactly $N - 2$ eigenvalues if $\alpha = \xi$ and $\beta = \pi$. Further, for any $0 \leq \alpha_1 < \alpha_2 < \xi \leq \alpha_3 < \alpha_4 < \pi$, and $0 < \beta_1 < \beta_2 \leq \pi$, we have that*

(i) *the eigenvalues of the SLPs $(\omega, \mathbf{S}_{\alpha_i, \beta_0})$ for any $\beta_0 \in (0, \pi)$, $i = 1, \dots, 4$, satisfy the following inequalities:*

$$\begin{aligned}\lambda_0(\alpha_2, \beta_0) &< \lambda_0(\alpha_1, \beta_0) < \lambda_0(\alpha_4, \beta_0) < \lambda_0(\alpha_3, \beta_0) < \lambda_1(\alpha_2, \beta_0) < \lambda_1(\alpha_1, \beta_0) \\ &< \lambda_1(\alpha_4, \beta_0) < \lambda_1(\alpha_3, \beta_0) < \dots < \lambda_{N-2}(\alpha_2, \beta_0) < \lambda_{N-2}(\alpha_1, \beta_0) < \\ &\lambda_{N-2}(\alpha_4, \beta_0) < \lambda_{N-2}(\alpha_3, \beta_0) < \lambda_{N-1}(\alpha_2, \beta_0) < \lambda_{N-1}(\alpha_1, \beta_0) < \lambda_{N-1}(\alpha_4, \beta_0), \\ \text{and in addition, } \lambda_{N-1}(\alpha_4, \beta_0) &< \lambda_{N-1}(\alpha_3, \beta_0) \text{ if } \alpha_3 \neq \xi;\end{aligned}$$

(ii) *similar results in (i) hold with $N - 2$ and $N - 1$ replaced by $N - 3$ and $N - 2$, respectively, in the case that $\beta_0 = \pi$;*

(iii) the eigenvalues of the SLPs $(\omega, \mathbf{S}_{\alpha_0, \beta_j})$ for any $\alpha_0 \in [0, \xi) \cup (\xi, \pi)$, $j = 1, 2$, satisfy the following inequalities:

$$\begin{aligned} \lambda_0(\alpha_0, \beta_1) &< \lambda_0(\alpha_0, \beta_2) < \lambda_1(\alpha_0, \beta_1) < \lambda_1(\alpha_0, \beta_2) < \cdots < \\ &< \lambda_{N-2}(\alpha_0, \beta_1) < \lambda_{N-2}(\alpha_0, \beta_2) < \lambda_{N-1}(\alpha_0, \beta_1), \end{aligned}$$

and in addition, $\lambda_{N-1}(\alpha_0, \beta_1) < \lambda_{N-1}(\alpha_0, \beta_2)$ if $\beta_2 \neq \pi$;

(iv) similar results in (iii) hold with $N - 2$ and $N - 1$ replaced by $N - 3$ and $N - 2$, respectively, in the case that $\alpha_0 = \xi$.

Proof. The number of eigenvalues of (ω, \mathbf{A}) in each case can be obtained by Lemma 2.4. Firstly, we show that (i) holds. Let $\beta_0 \in (0, \pi)$. By (i) of Corollary 4.2 in [22], the n -th eigenvalue functions $\lambda_n(\alpha, \beta_0)$ are strictly decreasing in $\alpha \in [0, \xi)$ or $\alpha \in (\xi, \pi)$ for all $0 \leq n \leq N - 1$. This implies that

$$\lambda_n(\alpha_2, \beta_0) < \lambda_n(\alpha_1, \beta_0), \quad 0 \leq n \leq N - 1, \quad \lambda_n(\alpha_4, \beta_0) < \lambda_n(\alpha_3, \beta_0), \quad 0 \leq n \leq N - 2, \quad (3.1)$$

and in addition, $\lambda_{N-1}(\alpha_4, \beta_0) < \lambda_{N-1}(\alpha_3, \beta_0)$ if $\alpha_3 \neq \xi$. Again by (i) of Corollary 4.2 in [22], $\lambda_n(\alpha, \beta_0)$, $0 \leq n \leq N - 1$, have the following asymptotic behaviors near 0 and ξ :

$$\begin{aligned} \lim_{\alpha \rightarrow \pi^-} \lambda_n(\alpha, \beta_0) &= \lambda_n(0, \beta_0), \quad 0 \leq n \leq N - 1, \\ \lim_{\alpha \rightarrow \xi^-} \lambda_0(\alpha, \beta_0) &= -\infty, \quad \lim_{\alpha \rightarrow \xi^-} \lambda_n(\alpha, \beta_0) = \lambda_{n-1}(\xi, \beta_0), \quad 1 \leq n \leq N - 1, \\ \lim_{\alpha \rightarrow \xi^+} \lambda_n(\alpha, \beta_0) &= \lambda_n(\xi, \beta_0), \quad 0 \leq n \leq N - 2, \quad \lim_{\alpha \rightarrow \xi^+} \lambda_{N-1}(\alpha, \beta_0) = +\infty. \end{aligned}$$

Thus,

$$\begin{aligned} \lambda_n(\alpha_1, \beta_0) &\leq \lambda_n(0, \beta_0) = \lim_{\alpha \rightarrow \pi^-} \lambda_n(\alpha, \beta_0) < \lambda_n(\alpha_4, \beta_0), \quad 0 \leq n \leq N - 1, \\ \lambda_n(\alpha_3, \beta_0) &\leq \lambda_n(\xi, \beta_0) = \lim_{\alpha \rightarrow \xi^-} \lambda_{n+1}(\alpha, \beta_0) < \lambda_{n+1}(\alpha_2, \beta_0), \quad 0 \leq n \leq N - 2, \end{aligned}$$

which together with (3.1) implies that (i) holds. See also Figure 3.1 for $N = 4$.

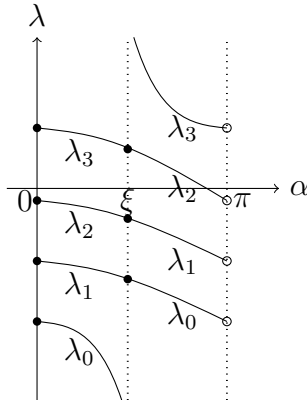


Figure 3.1. the n -th eigenvalue function of α .

The proof of assertion (ii) is similar to that for (i) by (iii) of Corollary 4.2 in [22].

Now, we show that (iii) holds. Let $\alpha_0 \in [0, \xi) \cup (\xi, \pi)$. By (ii) of Corollary 4.2 in [22], $\lambda_n(\alpha_0, \beta)$, $0 \leq n \leq N-1$, are strictly increasing in $\beta \in (0, \pi)$ for all $0 \leq n \leq N-1$. Thus,

$$\lambda_n(\alpha_0, \beta_1) < \lambda_n(\alpha_0, \beta_2), \quad 0 \leq n \leq N-2, \quad (3.2)$$

and in addition, $\lambda_{N-1}(\alpha_0, \beta_1) < \lambda_{N-1}(\alpha_0, \beta_2)$ if $\beta_2 \neq \pi$. Again by (ii) of Corollary 4.2 in [22], $\lambda_n(\alpha_0, \beta)$, $0 \leq n \leq N-1$, have the following asymptotic behaviors near 0 and π :

$$\begin{aligned} \lim_{\beta \rightarrow \pi^-} \lambda_n(\alpha_0, \beta) &= \lambda_n(\alpha_0, \pi), \quad 0 \leq n \leq N-2, \quad \lim_{\beta \rightarrow \pi^-} \lambda_{N-1}(\alpha_0, \beta) = +\infty, \\ \lim_{\beta \rightarrow 0^+} \lambda_0(\alpha_0, \beta) &= -\infty, \quad \lim_{\beta \rightarrow 0^+} \lambda_n(\alpha_0, \beta) = \lambda_{n-1}(\alpha_0, \pi), \quad 1 \leq n \leq N-1. \end{aligned}$$

Thus,

$$\lambda_n(\alpha_0, \beta_2) \leq \lambda_n(\alpha_0, \pi) = \lim_{\beta \rightarrow 0^+} \lambda_{n+1}(\alpha_0, \beta) < \lambda_{n+1}(\alpha_0, \beta_1), \quad 0 \leq n \leq N-2,$$

which together with (3.2) implies that (iii) holds. See also Figure 3.2 for $N = 4$.

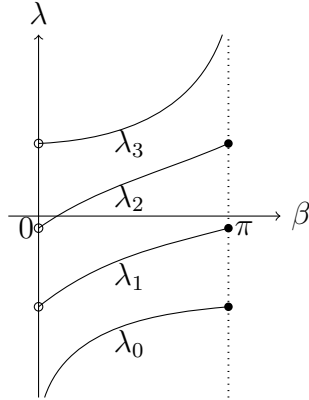


Figure 3.2. the n -th eigenvalue function of β .

The proof of assertion (iv) is similar to that for (iii) by (iv) of Corollary 4.2 in [22]. This completes the proof.

The following result is to compare eigenvalues for an arbitrarily separated BC with those for the BCs which are Dirichlet at an endpoint.

Corollary 3.1. *Fix a difference equation $\omega = (1/f, q, w)$ and a separated BC $\mathbf{S}_{\alpha_0, \beta_0}$. Then we have that*

(i) *for any $\alpha_0 \in (0, \xi)$ and $\beta_0 \in (0, \pi)$,*

$$\begin{aligned} \lambda_0(\alpha_0, \beta_0) &< \{\lambda_0(0, \beta_0), \lambda_0(\alpha_0, \pi)\} < \lambda_1(\alpha_0, \beta_0) < \{\lambda_1(0, \beta_0), \lambda_1(\alpha_0, \pi)\} < \cdots < \\ \lambda_{N-2}(\alpha_0, \beta_0) &< \{\lambda_{N-2}(0, \beta_0), \lambda_{N-2}(\alpha_0, \pi)\} < \lambda_{N-1}(\alpha_0, \beta_0) < \lambda_{N-1}(0, \beta_0); \end{aligned}$$

(ii) for any $\alpha_0 \in (\xi, \pi)$ and $\beta_0 \in (0, \pi)$,

$$\lambda_0(0, \beta_0) < \lambda_0(\alpha_0, \beta_0) < \{\lambda_1(0, \beta_0), \lambda_0(\alpha_0, \pi)\} < \lambda_1(\alpha_0, \beta_0) < \{\lambda_2(0, \beta_0), \lambda_1(\alpha_0, \pi)\} < \dots < \lambda_{N-2}(\alpha_0, \beta_0) < \{\lambda_{N-1}(0, \beta_0), \lambda_{N-2}(\alpha_0, \pi)\} < \lambda_{N-1}(\alpha_0, \beta_0);$$

(iii) for any $\alpha_0 = \xi$ and $\beta_0 \in (0, \pi)$,

$$\lambda_0(0, \beta_0) < \lambda_0(\alpha_0, \beta_0) < \{\lambda_1(0, \beta_0), \lambda_0(\alpha_0, \pi)\} < \lambda_1(\alpha_0, \beta_0) < \{\lambda_2(0, \beta_0), \lambda_1(\alpha_0, \pi)\} < \dots < \lambda_{N-3}(\alpha_0, \beta_0) < \{\lambda_{N-2}(0, \beta_0), \lambda_{N-3}(\alpha_0, \pi)\} < \lambda_{N-2}(\alpha_0, \beta_0) < \lambda_{N-1}(0, \beta_0);$$

(iv) for any $\alpha_0 \in (0, \xi)$ and $\beta_0 = \pi$,

$$\lambda_0(\alpha_0, \beta_0) < \lambda_0(0, \beta_0) < \lambda_1(\alpha_0, \beta_0) < \lambda_1(0, \beta_0) < \dots < \lambda_{N-2}(\alpha_0, \beta_0) < \lambda_{N-2}(0, \beta_0);$$

(v) for any $\alpha_0 \in (\xi, \pi)$ and $\beta_0 = \pi$,

$$\lambda_0(0, \beta_0) < \lambda_0(\alpha_0, \beta_0) < \lambda_1(0, \beta_0) < \lambda_1(\alpha_0, \beta_0) < \dots < \lambda_{N-2}(0, \beta_0) < \lambda_{N-2}(\alpha_0, \beta_0);$$

(vi) for any $\alpha_0 = \xi$ and $\beta_0 = \pi$,

$$\lambda_0(0, \beta_0) < \lambda_0(\alpha_0, \beta_0) < \dots < \lambda_{N-3}(0, \beta_0) < \lambda_{N-3}(\alpha_0, \beta_0) < \lambda_{N-2}(0, \beta_0);$$

(vii) for any $\alpha_0 = 0$ and $\beta_0 \in (0, \pi)$,

$$\lambda_0(\alpha_0, \beta_0) < \lambda_0(\alpha_0, \pi) < \dots < \lambda_{N-2}(\alpha_0, \beta_0) < \lambda_{N-2}(\alpha_0, \pi) < \lambda_{N-1}(\alpha_0, \beta_0),$$

where the notation $\{\lambda_0(0, \beta_0), \lambda_0(\alpha_0, \pi)\}$ means each of $\lambda_0(0, \beta_0)$ and $\lambda_0(\alpha_0, \pi)$, etc.

Proof. (i) and (iii), (i) and (iv), (ii), and (iii) of Theorem 3.1 imply that assertions (i)–(ii), (iii), (iv)–(vi), and (vii) hold, respectively. This completes the proof.

3.2. Inequalities among eigenvalues for different BCs in a natural loop

In this subsection, we shall establish inequalities among eigenvalues for different BCs in a natural loop (given in Lemma 2.7). We shall remark that inequalities among eigenvalues for different BCs in a natural loop will play an important role in establishing inequalities among eigenvalues for coupled BCs and those for some certain separated ones, and among eigenvalues for different coupled BCs in subsections 3.3 and 3.4.

Firstly, we shall establish inequalities among eigenvalues for different BCs in the natural loops $\mathcal{C}_{1,4,z,b_{21}}$ and $\mathcal{C}_{1,4,z,a_{12}}$, separately.

Theorem 3.2. Fix a difference equation $\omega = (1/f, q, w)$. Let

$$\mathbf{A}(a_{12}, b_{21}) := \begin{bmatrix} 1 & a_{12} & \bar{z} & 0 \\ 0 & z & b_{21} & 1 \end{bmatrix} \in \mathcal{O}_{1,4}^{\mathbb{C}}.$$

Then $(\omega, \mathbf{A}(a_{12}, b_{21}))$ has exactly N eigenvalues if $a_{12} \neq 1/f_0$ and exactly $N-1$ eigenvalues if $a_{12} = 1/f_0$; (ω, \mathbf{S}_1) has exactly N eigenvalues in any case; (ω, \mathbf{S}_2) has exactly $N-1$ eigenvalues if $a_{12} \neq 1/f_0$ and exactly $N-2$ eigenvalues if $a_{12} = 1/f_0$, where \mathbf{S}_1 and \mathbf{S}_2 are specified in Lemma 2.6. Further, for any $a_{12}^{(1)} < a_{12}^{(2)} \leq 1/f_0 < a_{12}^{(3)} < a_{12}^{(4)}$ and $b_{21}^{(1)} < b_{21}^{(2)}$, we have that

- (i) the eigenvalues $\lambda_n(a_{12}^{(i)})$ of the SLPs $(\omega, \mathbf{A}(a_{12}^{(i)}, b_{21}))$, $i = 1, \dots, 4$, and $\lambda_n(\mathbf{S}_1)$ of (ω, \mathbf{S}_1) satisfy the following inequalities:

$$\begin{aligned} \lambda_0(a_{12}^{(3)}) &\leq \lambda_0(a_{12}^{(4)}) \leq \lambda_0(\mathbf{S}_1) \leq \lambda_0(a_{12}^{(1)}) \leq \lambda_0(a_{12}^{(2)}) \leq \lambda_1(a_{12}^{(3)}) \leq \lambda_1(a_{12}^{(4)}) \leq \lambda_1(\mathbf{S}_1) \\ &\leq \lambda_1(a_{12}^{(1)}) \leq \lambda_1(a_{12}^{(2)}) \leq \dots \leq \lambda_{N-2}(a_{12}^{(3)}) \leq \lambda_{N-2}(a_{12}^{(4)}) \leq \lambda_{N-2}(\mathbf{S}_1) \leq \lambda_{N-2}(a_{12}^{(1)}) \\ &\leq \lambda_{N-2}(a_{12}^{(2)}) \leq \lambda_{N-1}(a_{12}^{(3)}) \leq \lambda_{N-1}(a_{12}^{(4)}) \leq \lambda_{N-1}(\mathbf{S}_1) \leq \lambda_{N-1}(a_{12}^{(1)}), \\ &\text{and in addition, } \lambda_{N-1}(a_{12}^{(1)}) \leq \lambda_{N-1}(a_{12}^{(2)}) \text{ if } a_{12}^{(2)} < 1/f_0; \end{aligned}$$

- (ii) the eigenvalues $\lambda_n(b_{21}^{(j)})$ of the SLPs $(\omega, \mathbf{A}(a_{12}, b_{21}^{(j)}))$, $j = 1, 2$, and $\lambda_n(\mathbf{S}_2)$ of (ω, \mathbf{S}_2) satisfy the following inequalities:

$$\begin{aligned} \lambda_0(b_{21}^{(1)}) &\leq \lambda_0(b_{21}^{(2)}) \leq \lambda_0(\mathbf{S}_2) \leq \lambda_1(b_{21}^{(1)}) \leq \lambda_1(b_{21}^{(2)}) \leq \lambda_1(\mathbf{S}_2) \leq \dots \leq \\ &\lambda_{N-3}(b_{21}^{(1)}) \leq \lambda_{N-3}(b_{21}^{(2)}) \leq \lambda_{N-3}(\mathbf{S}_2) \leq \lambda_{N-2}(b_{21}^{(1)}) \leq \lambda_{N-2}(b_{21}^{(2)}), \\ &\text{and in addition, } \lambda_{N-2}(b_{21}^{(2)}) \leq \lambda_{N-2}(\mathbf{S}_2) \leq \lambda_{N-1}(b_{21}^{(1)}) \leq \lambda_{N-1}(b_{21}^{(2)}) \text{ if } a_{12} \neq 1/f_0. \end{aligned}$$

Proof. The number of eigenvalues of $(\omega, \mathbf{A}(a_{12}, b_{21}))$, (ω, \mathbf{S}_1) , and (ω, \mathbf{S}_2) can be obtained by Lemma 2.4 and direct computations.

Let $\mathbf{A}(s)$ and $\mathcal{C}_{1,4,z,b_{21}} = \{\mathbf{A}(s) : s \in \mathbb{R}\} \cup \{\mathbf{S}_1\}$ be given as that in (i) of Lemma 2.7. Then $\mathbf{A}(a_{12}^{(i)}) = \mathbf{A}(a_{12}^{(i)}, b_{21})$, $i = 1, \dots, 4$. By (i)–(ii) of Theorem 4.1 in [22], the eigenvalue functions $\lambda_n(\mathbf{A}(s))$ are continuous and non-decreasing in $(-\infty, 1/f_0)$ and $(1/f_0, +\infty)$ for all $0 \leq n \leq N-1$. Thus, one gets that

$$\lambda_n(a_{12}^{(1)}) \leq \lambda_n(a_{12}^{(2)}), \quad 0 \leq n \leq N-2, \quad \lambda_n(a_{12}^{(3)}) \leq \lambda_n(a_{12}^{(4)}), \quad 0 \leq n \leq N-1, \quad (3.3)$$

and in addition, $\lambda_{N-1}(a_{12}^{(1)}) \leq \lambda_{N-1}(a_{12}^{(2)})$ if $a_{12}^{(2)} < 1/f_0$. By (iii) of Theorem 4.1 in [22], $\lambda_n(\mathbf{A}(s))$, $0 \leq n \leq N-1$, have asymptotic behaviors near $1/f_0$ as follows:

$$\begin{aligned} \lim_{s \rightarrow (1/f_0)^-} \lambda_n(\mathbf{A}(s)) &= \lambda_n(\mathbf{A}(1/f_0)), \quad 0 \leq n \leq N-2, \quad \lim_{s \rightarrow (1/f_0)^-} \lambda_{N-1}(\mathbf{A}(s)) = +\infty, \\ \lim_{s \rightarrow (1/f_0)^+} \lambda_0(\mathbf{A}(s)) &= -\infty, \quad \lim_{s \rightarrow (1/f_0)^+} \lambda_n(\mathbf{A}(s)) = \lambda_{n-1}(\mathbf{A}(1/f_0)), \quad 1 \leq n \leq N-1. \end{aligned}$$

Since $\mathcal{C}_{1,4,z,b_{21}} \setminus \{\mathbf{A}(1/f_0)\}$ is connected by Remark 2.1 and (ω, \mathbf{A}) has exactly N eigenvalues for each $\mathbf{A} \in \mathcal{C}_{1,4,z,b_{21}} \setminus \{\mathbf{A}(1/f_0)\}$, λ_n restricted in $\mathcal{C}_{1,4,z,b_{21}} \setminus \{\mathbf{A}(1/f_0)\}$ is continuous for each $0 \leq n \leq N-1$ by Lemma 2.5. This, together with Lemma 2.6, implies that $\lim_{s \rightarrow \pm\infty} \lambda_n(\mathbf{A}(s)) = \lambda_n(\mathbf{S}_1)$ for all $0 \leq n \leq N-1$. See Figure 3.3 for $N = 4$.

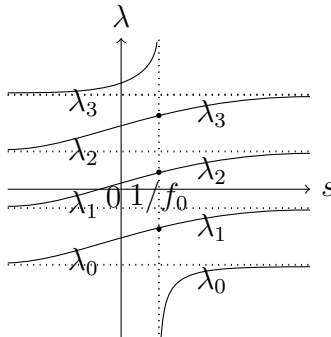


Figure 3.3. the n -th eigenvalue function of s .

Thus,

$$\begin{aligned} \lambda_n(a_{12}^{(4)}) &\leq \lambda_n(\mathbf{S}_1) \leq \lambda_n(a_{12}^{(1)}), \quad 0 \leq n \leq N-1, \\ \lambda_n(a_{12}^{(2)}) &\leq \lambda_n(\mathbf{A}(1/f_0)) = \lim_{s \rightarrow (1/f_0)^+} \lambda_{n+1}(\mathbf{A}(s)) \leq \lambda_{n+1}(a_{12}^{(3)}), \quad 0 \leq n \leq N-2. \end{aligned} \quad (3.4)$$

Hence (3.3)–(3.4) implies (i) holds.

Then we show that (ii) holds. Let $\hat{\mathbf{A}}(t)$ and $\mathcal{C}_{1,4,z,a_{12}} = \{\hat{\mathbf{A}}(t) : t \in \mathbb{R}\} \cup \{\mathbf{S}_2\}$ be given as that in (i) of Lemma 2.7. Then $\mathbf{A}(a_{12}, b_{21}^{(j)}) = \hat{\mathbf{A}}(b_{21}^{(j)})$, $j = 1, 2$.

Let $a_{12} \neq 1/f_0$. Then $\lambda_n(\hat{\mathbf{A}}(t))$ are continuous and non-decreasing in $t \in \mathbb{R}$ for all $0 \leq n \leq N-1$ by (i)–(ii) of Theorem 4.1 in [22]. Thus, for each $0 \leq n \leq N-1$, one has that

$$\lambda_n(b_{21}^{(1)}) \leq \lambda_n(b_{21}^{(2)}). \quad (3.5)$$

To see the limits of $\lambda_n(\hat{\mathbf{A}}(t))$ at $\pm\infty$, we notice that for $t \neq 0$,

$$\hat{\mathbf{A}}(t) = \begin{bmatrix} 1 & a_{12} & \bar{z} & 0 \\ 0 & z & t & 1 \end{bmatrix} = \begin{bmatrix} 1 & a_{12} - z\bar{z}/t & 0 & -\bar{z}/t \\ 0 & -z/t & -1 & -1/t \end{bmatrix}. \quad (3.6)$$

In the case that $a_{12} > 1/f_0$, direct computations show that $\hat{\mathbf{A}}(t) \in \mathcal{B}_{1,3r}^+$ if $t < 0$; $\hat{\mathbf{A}}(t) \in \mathcal{B}_{1,3}^-$ if $t > 0$; and $\mathbf{S}_2 \in \mathcal{B}_{1,3r}$, where

$$\begin{aligned} \mathcal{B}_{1,3r}^+ &:= \{\mathbf{A} \in \mathcal{O}_{1,3}^{\mathbb{C}} : a_{12} \geq 1/f_0, b_{22} \geq 0, (a_{12} - 1/f_0)b_{22} > |z|^2\}, \\ \mathcal{B}_{1,3}^- &:= \{\mathbf{A} \in \mathcal{O}_{1,3}^{\mathbb{C}} : (a_{12} - 1/f_0)b_{22} < |z|^2\}, \quad \mathbf{C} := \begin{bmatrix} 1 & 1/f_0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \\ \mathcal{B}_{1,3r} &:= \{\mathbf{A} \in \mathcal{O}_{1,3}^{\mathbb{C}} : (a_{12} - 1/f_0)b_{22} = |z|^2, a_{12} \geq 1/f_0, b_{22} \geq 0\} \setminus \{\mathbf{C}\}. \end{aligned}$$

Note that $\lim_{t \rightarrow \pm\infty} \hat{\mathbf{A}}(t) = \mathbf{S}_2$ by Lemma 2.6. Then, it follows from (iiia) of Theorem 4.3 in [22] that

$$\begin{aligned} \lim_{t \rightarrow -\infty} \lambda_0(\hat{\mathbf{A}}(t)) &= -\infty, \quad \lim_{t \rightarrow -\infty} \lambda_n(\hat{\mathbf{A}}(t)) = \lambda_{n-1}(\mathbf{S}_2), \quad 1 \leq n \leq N-1, \\ \lim_{t \rightarrow +\infty} \lambda_n(\hat{\mathbf{A}}(t)) &= \lambda_n(\mathbf{S}_2), \quad 0 \leq n \leq N-2, \quad \lim_{t \rightarrow +\infty} \lambda_{N-1}(\hat{\mathbf{A}}(t)) = +\infty. \end{aligned}$$

See Figure 3.4 for $N = 4$.

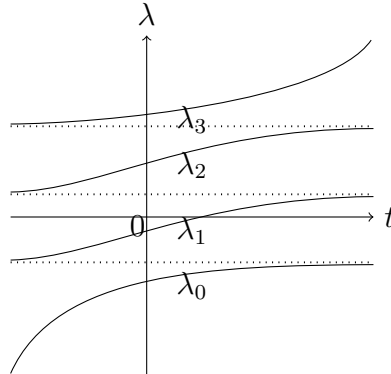


Figure 3.4. the n -th eigenvalue function of t .

Thus,

$$\lambda_n(b_{21}^{(2)}) \leq \lim_{t \rightarrow +\infty} \lambda_n(\hat{\mathbf{A}}(t)) = \lambda_n(\mathbf{S}_2) = \lim_{t \rightarrow -\infty} \lambda_{n+1}(\hat{\mathbf{A}}(t)) \leq \lambda_{n+1}(b_{21}^{(1)}), \quad (3.7)$$

for $0 \leq n \leq N-2$. (3.5) and (3.7) implies (ii) holds in the case that $a_{12} > 1/f_0$.

In the case that $a_{12} < 1/f_0$, direct computations imply that $\hat{\mathbf{A}}(t) \in \mathcal{B}_{1,3}^-$ if $t < 0$; $\hat{\mathbf{A}}(t) \in \mathcal{B}_{1,3l}^+$ if $t > 0$; and $\mathbf{S}_2 \in \mathcal{B}_{1,3l}$, where

$$\begin{aligned} \mathcal{B}_{1,3l}^+ &:= \{\mathbf{A} \in \mathcal{O}_{1,3}^{\mathbb{C}} : a_{12} \leq 1/f_0, b_{22} \leq 0, (a_{12} - 1/f_0)b_{22} > |z|^2\}, \\ \mathcal{B}_{1,3l} &:= \{\mathbf{A} \in \mathcal{O}_{1,3}^{\mathbb{C}} : (a_{12} - 1/f_0)b_{22} = |z|^2, a_{12} \leq 1/f_0, b_{22} \leq 0\} \setminus \{\mathbf{C}\}. \end{aligned}$$

By (iiib) of Theorem 4.3 in [22], similar arguments above yield that (ii) holds in this case.

Let $a_{12} = 1/f_0$. Then $\mathbf{S}_2 = \mathbf{C}$. Since $\mathcal{C}_{1,4,z,a_{12}} \setminus \{\mathbf{S}_2\}$ is connected by Remark 2.1 and (ω, \mathbf{A}) has exactly $N-1$ eigenvalues for each $\mathbf{A} \in \mathcal{C}_{1,4,z,a_{12}} \setminus \{\mathbf{S}_2\}$, by Lemma 2.5 the eigenvalue function λ_n is continuous and locally forms a continuous eigenvalue branch in $\mathcal{C}_{1,4,z,a_{12}} \setminus \{\mathbf{S}_2\}$ for each $0 \leq n \leq N-2$. By Theorem 4.6 in [23], $\lambda_n(\hat{\mathbf{A}}(t))$ is non-decreasing in $t \in \mathbb{R}$, and thus (3.5) holds for each $0 \leq n \leq N-2$. By (3.6) and direct computations, it follows that $\hat{\mathbf{A}}(t) \in \mathcal{B}_{1,3r}$ if $t < 0$; and $\hat{\mathbf{A}}(t) \in \mathcal{B}_{1,3l}$ if $t > 0$. In addition, $\lim_{t \rightarrow \pm\infty} \hat{\mathbf{A}}(t) = \mathbf{S}_2$ by Lemma 2.6. It follows from (iiic) of Theorem 4.3 in [22] that

$$\begin{aligned} \lim_{t \rightarrow -\infty} \lambda_0(\hat{\mathbf{A}}(t)) &= -\infty, \quad \lim_{t \rightarrow -\infty} \lambda_n(\hat{\mathbf{A}}(t)) = \lambda_{n-1}(\mathbf{S}_2), \quad 1 \leq n \leq N-2, \\ \lim_{t \rightarrow +\infty} \lambda_n(\hat{\mathbf{A}}(t)) &= \lambda_n(\mathbf{S}_2), \quad 0 \leq n \leq N-3, \quad \lim_{t \rightarrow +\infty} \lambda_{N-2}(\hat{\mathbf{A}}(t)) = +\infty. \end{aligned}$$

Thus, (3.7) holds for each $0 \leq n \leq N-3$. Hence, (ii) holds. The proof is complete.

Remark 3.1. The inequalities in Theorem 3.2 may not be strict. See the following example.

Example 3.1. Consider (1.1)–(1.2), where

$$f_0 = 1, \quad f_1 = 1, \quad f_2 = 1, \quad q_1 = q_2 = 0, \quad w_1 = w_2 = 1, \quad N = 2,$$

and

$$\mathbf{A}_1(s) := \begin{bmatrix} 1 & s & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \in \mathcal{O}_{1,4}^{\mathbb{C}}.$$

Then, by Lemma 2.3,

$$\Gamma(\lambda) = -(s-1)\lambda^2 + 2(s-2)\lambda.$$

Thus, for each $s \in (-\infty, 1) \cup (1, \infty)$, there are exactly two eigenvalues for $\mathbf{A}_1(s)$ and exactly one eigenvalue for $\mathbf{A}_1(1)$:

$$\lambda_0(s) = \begin{cases} 0 & \text{if } s \leq 1, \\ 2(s-2)/(s-1) & \text{if } 1 < s \leq 2, \\ 0 & \text{if } s > 2, \end{cases} \quad \lambda_1(s) = \begin{cases} 2(s-2)/(s-1) & \text{if } s < 1, \\ 0 & \text{if } 1 < s \leq 2, \\ 2(s-2)/(s-1) & \text{if } s > 2. \end{cases}$$

Note that

$$\mathbf{S}_1 = \lim_{s \rightarrow +\infty} \mathbf{A}_1(s) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

It is easy to see that there are exactly two eigenvalues for \mathbf{S}_1 , and they are 0 and 2. See Figure 3.5.

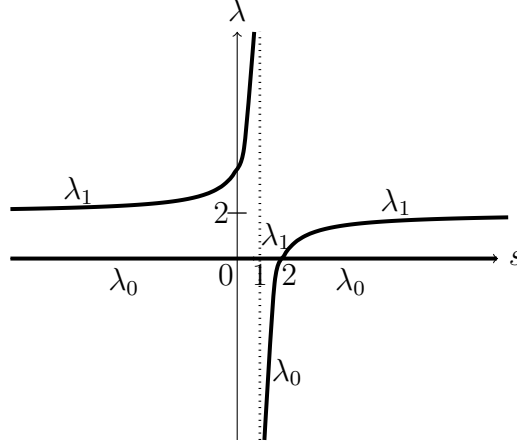


Figure 3.5. the n -th eigenvalue function of s in Example 3.1.

Then $\lambda_0(\mathbf{S}_1) = \lambda_0(\mathbf{A}_1(s)) < \lambda_1(\mathbf{S}_1) < \lambda_1(\mathbf{A}_1(s))$ for $s < 1$; $\lambda_0(\mathbf{S}_1) = \lambda_0(\mathbf{A}_1(s)) < \lambda_1(\mathbf{S}_1)$ for $s = 1$; $\lambda_0(\mathbf{A}_1(s)) < \lambda_0(\mathbf{S}_1) = \lambda_1(\mathbf{A}_1(s)) < \lambda_1(\mathbf{S}_1)$ for $1 < s < 2$; $\lambda_0(\mathbf{A}_1(s)) = \lambda_0(\mathbf{S}_1) = \lambda_1(\mathbf{A}_1(s)) < \lambda_1(\mathbf{S}_1)$ for $s = 2$; and $\lambda_0(\mathbf{A}_1(s)) = \lambda_0(\mathbf{S}_1) < \lambda_1(\mathbf{A}_1(s)) < \lambda_1(\mathbf{S}_1)$ for $s > 2$.

Secondly, we shall establish inequalities among eigenvalues for different BCs in the natural loops $\mathcal{C}_{2,4,z,b_{21}}$ and $\mathcal{C}_{2,4,z,a_{11}}$, separately.

Theorem 3.3. Fix a difference equation $\omega = (1/f, q, w)$. Let

$$\mathbf{A}(a_{11}, b_{21}) := \begin{bmatrix} a_{11} & -1 & \bar{z} & 0 \\ z & 0 & b_{21} & 1 \end{bmatrix} \in \mathcal{O}_{2,4}^{\mathbb{C}}.$$

Then similar results in Theorem 3.2 hold with a_{12} , $a_{12}^{(i)}$, $1/f_0$, \mathbf{S}_1 , and \mathbf{S}_2 replaced by a_{11} , $a_{11}^{(i)}$, $-f_0$, \mathbf{S}_3 , and \mathbf{S}_4 , separately, where $i = 1, \dots, 4$, and \mathbf{S}_3 and \mathbf{S}_4 are specified in Lemma 2.6.

Proof. By a similar method to that used in the proof of Theorem 3.2, one can show that Theorem 3.3 holds with the help of Theorems 4.2 and 4.4 in [22].

Thirdly, we shall establish inequalities among eigenvalues for different BCs in the natural loops $\mathcal{C}_{2,3,z,b_{22}}$ and $\mathcal{C}_{2,3,z,a_{11}}$, separately. We shall remark that here we only give the inequalities in the case that $z \neq 0$ since we shall apply Theorem 3.4 to coupled BCs, which satisfy that $z \neq 0$. One can establish the inequalities in the case that $z = 0$ with a similar method.

Theorem 3.4. Fix a difference equation $\omega = (1/f, q, w)$. Let

$$\mathbf{A}(a_{11}, b_{22}) := \begin{bmatrix} a_{11} & -1 & 0 & \bar{z} \\ z & 0 & -1 & b_{22} \end{bmatrix} \in \mathcal{O}_{2,3}^{\mathbb{C}},$$

where $z \neq 0$. Then $(\boldsymbol{\omega}, \mathbf{A}(a_{11}, b_{22}))$ has exactly N eigenvalues if $b_{22}(a_{11} + f_0) \neq |z|^2$, and exactly $N - 1$ eigenvalues if $b_{22}(a_{11} + f_0) = |z|^2$; $(\boldsymbol{\omega}, \mathbf{S}_5)$ has exactly N eigenvalues if $b_{22} \neq 0$, and exactly $N - 1$ eigenvalues if $b_{22} = 0$; $(\boldsymbol{\omega}, \mathbf{S}_6)$ has exactly N eigenvalues if $a_{11} + f_0 \neq 0$, and exactly $N - 1$ eigenvalues if $a_{11} + f_0 = 0$, where \mathbf{S}_5 and \mathbf{S}_6 are specified in Lemma 2.6. Further, we have that

- (i) in the case that $b_{22} = 0$, for any $a_{11}^{(1)} < a_{11}^{(2)}$, the eigenvalues $\lambda_n(a_{11}^{(i)})$ of $(\boldsymbol{\omega}, \mathbf{A}(a_{11}^{(i)}, b_{22}))$, $i = 1, 2$, and $\lambda_n(\mathbf{S}_5)$ of $(\boldsymbol{\omega}, \mathbf{S}_5)$ satisfy the following inequalities:

$$\begin{aligned} \lambda_0(a_{11}^{(1)}) &\leq \lambda_0(a_{11}^{(2)}) \leq \lambda_0(\mathbf{S}_5) \leq \lambda_1(a_{11}^{(1)}) \leq \lambda_1(a_{11}^{(2)}) \leq \lambda_1(\mathbf{S}_5) \leq \cdots \leq \\ \lambda_{N-2}(a_{11}^{(1)}) &\leq \lambda_{N-2}(a_{11}^{(2)}) \leq \lambda_{N-2}(\mathbf{S}_5) \leq \lambda_{N-1}(a_{11}^{(1)}) \leq \lambda_{N-1}(a_{11}^{(2)}); \end{aligned} \quad (3.8)$$

- (ii) in the case that $b_{22} \neq 0$, for any $a_{11}^{(1)} < a_{11}^{(2)} \leq |z|^2/b_{22} - f_0$ and $|z|^2/b_{22} - f_0 < a_{11}^{(3)} < a_{11}^{(4)}$, the eigenvalues $\lambda_n(a_{11}^{(i)})$ of $(\boldsymbol{\omega}, \mathbf{A}(a_{11}^{(i)}, b_{22}))$, $i = 1, \dots, 4$, and $\lambda_n(\mathbf{S}_5)$ of $(\boldsymbol{\omega}, \mathbf{S}_5)$ satisfy the following inequalities:

$$\begin{aligned} \lambda_0(a_{11}^{(3)}) &\leq \lambda_0(a_{11}^{(4)}) \leq \lambda_0(\mathbf{S}_5) \leq \lambda_0(a_{11}^{(1)}) \leq \lambda_0(a_{11}^{(2)}) \leq \\ \lambda_1(a_{11}^{(3)}) &\leq \lambda_1(a_{11}^{(4)}) \leq \lambda_1(\mathbf{S}_5) \leq \lambda_1(a_{11}^{(1)}) \leq \lambda_1(a_{11}^{(2)}) \\ &\leq \cdots \leq \lambda_{N-2}(a_{11}^{(3)}) \leq \lambda_{N-2}(a_{11}^{(4)}) \leq \lambda_{N-2}(\mathbf{S}_5) \leq \lambda_{N-2}(a_{11}^{(1)}) \leq \\ \lambda_{N-2}(a_{11}^{(2)}) &\leq \lambda_{N-1}(a_{11}^{(3)}) \leq \lambda_{N-1}(a_{11}^{(4)}) \leq \lambda_{N-1}(\mathbf{S}_5) \leq \lambda_{N-1}(a_{11}^{(1)}), \end{aligned} \quad (3.9)$$

and in addition, $\lambda_{N-1}(a_{11}^{(1)}) \leq \lambda_{N-1}(a_{11}^{(2)})$ if $a_{11}^{(2)} < |z|^2/b_{22} - f_0$;

- (iii) in the case that $a_{11} + f_0 = 0$, for any $b_{22}^{(1)} < b_{22}^{(2)}$, the eigenvalues $\lambda_n(b_{22}^{(i)})$ of $(\boldsymbol{\omega}, \mathbf{A}(a_{11}, b_{22}^{(i)}))$ and $\lambda_n(\mathbf{S}_6)$ of $(\boldsymbol{\omega}, \mathbf{S}_6)$ satisfy (3.8) with $a_{11}^{(i)}$ and \mathbf{S}_5 replaced by $b_{22}^{(i)}$ and \mathbf{S}_6 , separately, where $i = 1, 2$;

- (iv) in the case that $a_{11} + f_0 \neq 0$, for any $b_{22}^{(1)} < b_{22}^{(2)} \leq |z|^2/(a_{11} + f_0)$ and $|z|^2/(a_{11} + f_0) < b_{22}^{(3)} < b_{22}^{(4)}$, the eigenvalues $\lambda_n(b_{22}^{(i)})$ of $(\boldsymbol{\omega}, \mathbf{A}(a_{11}, b_{22}^{(i)}))$ and $\lambda_n(\mathbf{S}_6)$ of $(\boldsymbol{\omega}, \mathbf{S}_6)$ satisfy (3.9) with $a_{11}^{(i)}$ and \mathbf{S}_5 replaced by $b_{22}^{(i)}$ and \mathbf{S}_6 , separately, where $i = 1, \dots, 4$, and in addition, $\lambda_{N-1}(b_{22}^{(1)}) \leq \lambda_{N-1}(b_{22}^{(2)})$ if $b_{22}^{(2)} < |z|^2/(a_{11} + f_0)$.

Proof. By a similar method to that used in the proof of (ii) in Theorem 3.2, one can show that (i) holds with the help of Theorems 4.3–4.4 of [22]; (iii) holds with the help of Theorems 4.2 and 4.4 of [22]. By a similar method to that used in the proof of (i) in Theorem 3.2, one gets that (ii) and (iv) hold with the help of Theorem 4.4 of [22]. This completes the proof.

Fourthly, we shall establish inequalities among eigenvalues for different BCs in the natural loops $\mathcal{C}_{1,3,z,b_{22}}$ and $\mathcal{C}_{1,3,z,a_{12}}$ with $z \neq 0$, separately.

Theorem 3.5. Fix a difference equation $\omega = (1/f, q, w)$. Let

$$\mathbf{A}(a_{12}, b_{22}) := \begin{bmatrix} 1 & a_{12} & 0 & \bar{z} \\ 0 & z & -1 & b_{22} \end{bmatrix} \in \mathcal{O}_{1,3}^{\mathbb{C}},$$

where $z \neq 0$. Then similar results in Theorem 3.4 hold for a_{11} , $a_{11}^{(i)}$, $a_{11} + f_0$, $|z|^2/b_{22} - f_0$, \mathbf{S}_5 , and \mathbf{S}_6 replaced by a_{12} , $a_{12}^{(i)}$, $a_{12} - 1/f_0$, $|z|^2/b_{22} + 1/f_0$, \mathbf{S}_7 , and \mathbf{S}_8 , separately, where $i = 1, \dots, 4$, and \mathbf{S}_7 and \mathbf{S}_8 are specified in Lemma 2.6.

Proof. By a similar method to that used in the proof of (ii) in Theorem 3.2, one can show that (i) holds with the help of Theorems 4.3–4.4 of [22], and (iii) holds with the help of Theorems 4.1 and 4.4 of [22]. By a similar method to that used in the proof of (i) in Theorem 3.2, one gets that (ii) and (iv) hold with the help of Theorem 4.3 of [22]. This completes the proof.

3.3. Inequalities among eigenvalues for coupled BCs and those for some certain separated ones

In this subsection, we shall first establish inequalities among eigenvalues for a coupled BC and those for some certain separated ones applying Theorems 3.2–3.5. Then, for a fixed $K \in SL(2, \mathbb{R})$ and $\gamma \in (-\pi, 0) \cup (0, \pi)$, we shall compare eigenvalues for $[K| - I]$, those for $[e^{i\gamma}K| - I]$, and those for $[-K| - I]$. Combining the above two parts, we shall establish inequalities among eigenvalues for three coupled BCs and those for some certain separated ones, which generalize the main result of [18].

Firstly, we shall establish inequalities among eigenvalues for a coupled BC and those for some certain separated ones in the next two theorems. Set $\lambda_n(e^{i\gamma}K) := \lambda_n([e^{i\gamma}K| - I])$ for brevity.

Theorem 3.6. Fix a difference equation $\omega = (1/f, q, w)$. Let $\mathbf{A} = [e^{i\gamma}K| - I]$, where $K \in SL(2, \mathbb{R})$ and $\gamma \in (-\pi, \pi]$. Then (ω, \mathbf{A}) has exactly N eigenvalues if $k_{11} - f_0k_{12} \neq 0$, and exactly $N - 1$ eigenvalues if $k_{11} - f_0k_{12} = 0$; (ω, \mathbf{T}_K) has exactly N eigenvalues if $k_{11} \neq 0$, and exactly $N - 1$ eigenvalues if $k_{11} = 0$; (ω, \mathbf{U}_K) has exactly $N - 1$ eigenvalues if $k_{11} - f_0k_{12} \neq 0$, and exactly $N - 2$ eigenvalues if $k_{11} - f_0k_{12} = 0$, where

$$\mathbf{T}_K := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -k_{21} & k_{11} \end{bmatrix} \text{ and } \mathbf{U}_K := \begin{bmatrix} k_{11} & k_{12} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Furthermore, we have that

(i) the eigenvalues of (ω, \mathbf{A}) and (ω, \mathbf{T}_K) satisfy the following inequalities:

$$\begin{aligned} \lambda_0(\mathbf{T}_K) &\leq \lambda_0(e^{i\gamma}K) \leq \lambda_1(\mathbf{T}_K) \leq \lambda_1(e^{i\gamma}K) \\ &\leq \dots \leq \lambda_{N-1}(\mathbf{T}_K) \leq \lambda_{N-1}(e^{i\gamma}K) \end{aligned} \tag{3.10}$$

in the case that $(k_{11} - f_0k_{12})k_{11}f_0 > 0$;

$$\begin{aligned}\lambda_0(e^{i\gamma}K) &\leq \lambda_0(\mathbf{T}_K) \leq \lambda_1(e^{i\gamma}K) \leq \lambda_1(\mathbf{T}_K) \\ &\leq \cdots \leq \lambda_{N-1}(e^{i\gamma}K) \leq \lambda_{N-1}(\mathbf{T}_K)\end{aligned}\tag{3.11}$$

in the case that $(k_{11} - f_0 k_{12})k_{11}f_0 < 0$;

$$\begin{aligned}\lambda_0(\mathbf{T}_K) &\leq \lambda_0(e^{i\gamma}K) \leq \lambda_1(\mathbf{T}_K) \leq \lambda_1(e^{i\gamma}K) \\ &\leq \cdots \leq \lambda_{N-2}(\mathbf{T}_K) \leq \lambda_{N-2}(e^{i\gamma}K) \leq \lambda_{N-1}(\mathbf{T}_K)\end{aligned}\tag{3.12}$$

in the case that $k_{11} - f_0 k_{12} = 0$;

$$\begin{aligned}\lambda_0(e^{i\gamma}K) &\leq \lambda_0(\mathbf{T}_K) \leq \lambda_1(e^{i\gamma}K) \leq \lambda_1(\mathbf{T}_K) \\ &\leq \cdots \leq \lambda_{N-2}(e^{i\gamma}K) \leq \lambda_{N-2}(\mathbf{T}_K) \leq \lambda_{N-1}(e^{i\gamma}K)\end{aligned}\tag{3.13}$$

in the case that $k_{11} = 0$;

(ii) the eigenvalues of $(\boldsymbol{\omega}, \mathbf{A})$ and $(\boldsymbol{\omega}, \mathbf{U}_K)$ satisfy the following inequalities:

$$\begin{aligned}\lambda_0(e^{i\gamma}K) &\leq \lambda_0(\mathbf{U}_K) \leq \lambda_1(e^{i\gamma}K) \leq \lambda_1(\mathbf{U}_K) \\ &\leq \cdots \leq \lambda_{N-2}(e^{i\gamma}K) \leq \lambda_{N-2}(\mathbf{U}_K) \leq \lambda_{N-1}(e^{i\gamma}K)\end{aligned}\tag{3.14}$$

in the case that $k_{11} - f_0 k_{12} \neq 0$;

$$\begin{aligned}\lambda_0(e^{i\gamma}K) &\leq \lambda_0(\mathbf{U}_K) \leq \lambda_1(e^{i\gamma}K) \leq \lambda_1(\mathbf{U}_K) \\ &\leq \cdots \leq \lambda_{N-3}(e^{i\gamma}K) \leq \lambda_{N-3}(\mathbf{U}_K) \leq \lambda_{N-2}(e^{i\gamma}K)\end{aligned}\tag{3.15}$$

in the case that $k_{11} - f_0 k_{12} = 0$.

Proof. The number of eigenvalues of $(\boldsymbol{\omega}, \mathbf{A})$, $(\boldsymbol{\omega}, \mathbf{T}_K)$, and $(\boldsymbol{\omega}, \mathbf{U}_K)$ in each case can be obtained by Lemma 2.4 and direct computations. Let $k_{11} \neq 0$. Since $\det K = 1$,

$$\mathbf{A} = [e^{i\gamma}K| - I] = \begin{bmatrix} 1 & k_{12}/k_{11} & -e^{-i\gamma}/k_{11} & 0 \\ -e^{i\gamma}k_{21} & -e^{i\gamma}k_{22} & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a_{12} & \bar{z} & 0 \\ 0 & z & b_{21} & 1 \end{bmatrix} \in \mathcal{O}_{1,4}^{\mathbb{C}},$$

where $a_{12} := k_{12}/k_{11}$, $b_{21} := -k_{21}/k_{11}$, and $z := -e^{i\gamma}/k_{11}$. Then by (i) of Lemma 2.7, $\mathbf{A} \in \mathcal{C}_{1,4,z,b_{21}} \cap \mathcal{C}_{1,4,z,a_{12}}$, and the corresponding LBCs satisfy that

$$\begin{aligned}\mathbf{S}_1 &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & b_{21} & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -k_{21} & k_{11} \end{bmatrix} = \mathbf{T}_K, \\ \mathbf{S}_2 &= \begin{bmatrix} 1 & a_{12} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \mathbf{U}_K.\end{aligned}$$

Note that $(k_{11} - f_0 k_{12})k_{11}f_0 > 0$, $(k_{11} - f_0 k_{12})k_{11}f_0 < 0$, $k_{11} - f_0 k_{12} = 0$, and $k_{11} - f_0 k_{12} \neq 0$ are equivalent to $a_{12} < 1/f_0$, $a_{12} > 1/f_0$, $a_{12} = 1/f_0$, and $a_{12} \neq 1/f_0$, respectively. Therefore, by Theorem 3.2, one gets that $(k_{11} - f_0 k_{12})k_{11}f_0 > 0$ implies (3.10); $(k_{11} - f_0 k_{12})k_{11}f_0 < 0$ implies (3.11); $k_{11} - f_0 k_{12} = 0$ implies (3.12) and (3.15); $k_{11} - f_0 k_{12} \neq 0$ implies (3.14).

Let $k_{11} = 0$. Now we show that (3.13)–(3.14) hold in this case. Since $k_{11} = 0$, $-k_{12}k_{21} = 1$. Denote

$$K_\epsilon := \begin{pmatrix} \epsilon & k_{12} \\ (-1 + \epsilon k_{22})/k_{12} & k_{22} \end{pmatrix} \in SL(2, \mathbb{R}), \quad \epsilon \in \mathbb{R}.$$

Then $\lim_{\epsilon \rightarrow 0} K_\epsilon = K$. By the definition of \mathbf{T}_K and \mathbf{U}_K , we see that

$$\mathbf{T}_{K_\epsilon} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 - \epsilon k_{22} & \epsilon k_{12} \end{bmatrix} \text{ and } \mathbf{U}_{K_\epsilon} = \begin{bmatrix} \epsilon & k_{12} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Then $[e^{i\gamma} K_\epsilon] - I \rightarrow [e^{i\gamma} K] - I$, $\mathbf{T}_{K_\epsilon} \rightarrow \mathbf{T}_K$, $\mathbf{U}_{K_\epsilon} \rightarrow \mathbf{U}_K$, as $\epsilon \rightarrow 0$.

Since $k_{12} \neq 0$, one can choose a sufficiently small $\epsilon_1 > 0$ such that $\epsilon - f_0 k_{12} \neq 0$, where $0 \leq \epsilon \leq \epsilon_1$. Thus, by Lemma 2.4, there are exactly N eigenvalues for each $[e^{i\gamma} K_\epsilon] - I$, $0 \leq \epsilon \leq \epsilon_1$, and by Lemma 2.5, $\lambda_n(e^{i\gamma} K_\epsilon)$ is continuous in $\epsilon \in [0, \epsilon_1]$, which implies that

$$\lambda_n(e^{i\gamma} K_\epsilon) \rightarrow \lambda_n(e^{i\gamma} K), \text{ as } \epsilon \rightarrow 0^+, \quad 0 \leq n \leq N - 1. \quad (3.16)$$

Suppose that $k_{12} < 0$. By Lemma 2.4, $(\omega, \mathbf{U}_{K_\epsilon})$ has exactly $N - 1$ eigenvalues for each $\epsilon \in [0, \epsilon_1]$. Thus by Lemma 2.5, $\lambda_n(\mathbf{U}_{K_\epsilon})$ is continuous in $\epsilon \in [0, \epsilon_1]$, which implies that

$$\lambda_n(\mathbf{U}_{K_\epsilon}) \rightarrow \lambda_n(\mathbf{U}_K), \text{ as } \epsilon \rightarrow 0^+, \quad 0 \leq n \leq N - 2. \quad (3.17)$$

If $f_0 > 0$, then $\mathbf{T}_K \in \mathcal{B}_{2,3r}$ and $\mathbf{T}_{K_\epsilon} \in \mathcal{B}_{2,3r}^+$, where $\epsilon \in (0, \epsilon_1]$ and

$$\begin{aligned} \mathcal{B}_{2,3r} &:= \{\mathbf{A} \in \mathcal{O}_{2,3}^{\mathbb{C}} : (a_{11} + f_0)b_{22} = |z|^2, a_{11} + f_0 \geq 0, b_{22} \geq 0\} \setminus \{\mathbf{C}\}, \\ \mathcal{B}_{2,3r}^+ &:= \{\mathbf{A} \in \mathcal{O}_{2,3}^{\mathbb{C}} : a_{11} \geq -f_0, b_{22} \geq 0, (a_{11} + f_0)b_{22} > |z|^2\}. \end{aligned}$$

Note that ϵ_1 can be chosen such that $1 - \epsilon k_{22} > 0$ for any $0 < \epsilon \leq \epsilon_1$. By Theorem 4.4 in [22],

$$\lambda_0(\mathbf{T}_{K_\epsilon}) \rightarrow -\infty, \quad \lambda_n(\mathbf{T}_{K_\epsilon}) \rightarrow \lambda_{n-1}(\mathbf{T}_K), \text{ as } \epsilon \rightarrow 0^+, \quad 1 \leq n \leq N - 1. \quad (3.18)$$

If $f_0 < 0$, then $\mathbf{T}_K \in \mathcal{B}_{2,3l}$ and $\mathbf{T}_{K_\epsilon} \in \mathcal{B}_{2,3}^-$, where $\epsilon \in (0, \epsilon_1]$ and

$$\begin{aligned} \mathcal{B}_{2,3l} &:= \{\mathbf{A} \in \mathcal{O}_{2,3}^{\mathbb{C}} : (a_{11} + f_0)b_{22} = |z|^2, a_{11} + f_0 \leq 0, b_{22} \leq 0\} \setminus \{\mathbf{C}\}, \\ \mathcal{B}_{2,3}^- &:= \{\mathbf{A} \in \mathcal{O}_{2,3}^{\mathbb{C}} : (a_{11} + f_0)b_{22} < |z|^2\}. \end{aligned}$$

By Theorem 4.4 in [22], (3.18) holds.

Since $(\epsilon - f_0 k_{12})f_0 \epsilon > 0$, where $0 < \epsilon \leq \epsilon_1$, by (3.10) and (3.14) for $[e^{i\gamma} K_\epsilon] - I$,

$$\begin{aligned} \lambda_0(\mathbf{T}_{K_\epsilon}) &\leq \lambda_0(e^{i\gamma} K_\epsilon) \leq \{\lambda_1(\mathbf{T}_{K_\epsilon}), \lambda_0(\mathbf{U}_{K_\epsilon})\} \leq \lambda_1(e^{i\gamma} K_\epsilon) \leq \{\lambda_2(\mathbf{T}_{K_\epsilon}), \\ \lambda_1(\mathbf{U}_{K_\epsilon})\} &\leq \cdots \leq \lambda_{N-2}(e^{i\gamma} K_\epsilon) \leq \{\lambda_{N-1}(\mathbf{T}_{K_\epsilon}), \lambda_{N-2}(\mathbf{U}_{K_\epsilon})\} \leq \lambda_{N-1}(e^{i\gamma} K_\epsilon). \end{aligned} \quad (3.19)$$

Let $\epsilon \rightarrow 0^+$ in (3.19), it follows from (3.16)–(3.18) that (3.13)–(3.14) hold for $[e^{i\gamma} K] - I$.

Suppose that $k_{12} > 0$. With a similar method to that used in the case that $k_{12} < 0$, one can show that (3.13)–(3.14) hold for $[e^{i\gamma} K] - I$. The proof is complete.

Theorem 3.7. Fix a difference equation $\omega = (1/f, q, w)$. Let $\mathbf{A} = [e^{i\gamma}K] - I$, where $K \in SL(2, \mathbb{R})$ and $\gamma \in (-\pi, \pi]$. Then (ω, \mathbf{S}_K) has exactly N eigenvalues if $k_{12} \neq 0$, and exactly $N - 1$ eigenvalues if $k_{12} = 0$; (ω, \mathbf{V}_K) has exactly N eigenvalues if $f_0 k_{22} - k_{21} \neq 0$, and exactly $N - 1$ eigenvalues if $f_0 k_{22} - k_{21} = 0$, where

$$\mathbf{S}_K := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -k_{22} & k_{12} \end{bmatrix} \text{ and } \mathbf{V}_K := \begin{bmatrix} k_{21} & k_{22} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Furthermore, we have that

(i) the eigenvalues of (ω, \mathbf{A}) and (ω, \mathbf{S}_K) satisfy the following inequalities:

$$\begin{aligned} \lambda_0(e^{i\gamma}K) &\leq \lambda_0(\mathbf{S}_K) \leq \lambda_1(e^{i\gamma}K) \leq \lambda_1(\mathbf{S}_K) \\ &\leq \cdots \leq \lambda_{N-2}(e^{i\gamma}K) \leq \lambda_{N-2}(\mathbf{S}_K) \leq \lambda_{N-1}(e^{i\gamma}K) \end{aligned} \quad (3.20)$$

in the case that $k_{12} = 0$;

$$\begin{aligned} \lambda_0(\mathbf{S}_K) &\leq \lambda_0(e^{i\gamma}K) \leq \lambda_1(\mathbf{S}_K) \leq \lambda_1(e^{i\gamma}K) \\ &\leq \cdots \leq \lambda_{N-1}(\mathbf{S}_K) \leq \lambda_{N-1}(e^{i\gamma}K) \end{aligned} \quad (3.21)$$

in the case that $(k_{11} - f_0 k_{12})k_{12} > 0$;

$$\begin{aligned} \lambda_0(e^{i\gamma}K) &\leq \lambda_0(\mathbf{S}_K) \leq \lambda_1(e^{i\gamma}K) \leq \lambda_1(\mathbf{S}_K) \\ &\leq \cdots \leq \lambda_{N-1}(e^{i\gamma}K) \leq \lambda_{N-1}(\mathbf{S}_K) \end{aligned} \quad (3.22)$$

in the case that $(k_{11} - f_0 k_{12})k_{12} < 0$;

$$\begin{aligned} \lambda_0(\mathbf{S}_K) &\leq \lambda_0(e^{i\gamma}K) \leq \lambda_1(\mathbf{S}_K) \leq \lambda_1(e^{i\gamma}K) \\ &\leq \cdots \leq \lambda_{N-2}(\mathbf{S}_K) \leq \lambda_{N-2}(e^{i\gamma}K) \leq \lambda_{N-1}(\mathbf{S}_K) \end{aligned} \quad (3.23)$$

in the case that $k_{11} - f_0 k_{12} = 0$;

(ii) the eigenvalues of (ω, \mathbf{A}) and (ω, \mathbf{V}_K) satisfy the following inequalities:

$$\begin{aligned} \lambda_0(e^{i\gamma}K) &\leq \lambda_0(\mathbf{V}_K) \leq \lambda_1(e^{i\gamma}K) \leq \lambda_1(\mathbf{V}_K) \\ &\leq \cdots \leq \lambda_{N-2}(e^{i\gamma}K) \leq \lambda_{N-2}(\mathbf{V}_K) \leq \lambda_{N-1}(e^{i\gamma}K) \end{aligned} \quad (3.24)$$

in the case that $f_0 k_{22} - k_{21} = 0$;

$$\begin{aligned} \lambda_0(\mathbf{V}_K) &\leq \lambda_0(e^{i\gamma}K) \leq \lambda_1(\mathbf{V}_K) \leq \lambda_1(e^{i\gamma}K) \\ &\leq \cdots \leq \lambda_{N-1}(\mathbf{V}_K) \leq \lambda_{N-1}(e^{i\gamma}K) \end{aligned} \quad (3.25)$$

in the case that $(k_{11} - f_0 k_{12})(f_0 k_{22} - k_{21}) > 0$;

$$\begin{aligned} \lambda_0(e^{i\gamma}K) &\leq \lambda_0(\mathbf{V}_K) \leq \lambda_1(e^{i\gamma}K) \leq \lambda_1(\mathbf{V}_K) \\ &\leq \cdots \leq \lambda_{N-1}(e^{i\gamma}K) \leq \lambda_{N-1}(\mathbf{V}_K) \end{aligned} \quad (3.26)$$

in the case that $(k_{11} - f_0 k_{12})(f_0 k_{22} - k_{21}) < 0$;

$$\begin{aligned} \lambda_0(\mathbf{V}_K) &\leq \lambda_0(e^{i\gamma}K) \leq \lambda_1(\mathbf{V}_K) \leq \lambda_1(e^{i\gamma}K) \\ &\leq \cdots \leq \lambda_{N-2}(\mathbf{V}_K) \leq \lambda_{N-2}(e^{i\gamma}K) \leq \lambda_{N-1}(\mathbf{V}_K) \end{aligned} \quad (3.27)$$

in the case that $k_{11} - f_0 k_{12} = 0$.

Proof. The number of eigenvalues of (ω, \mathbf{A}) , (ω, \mathbf{S}_K) , and (ω, \mathbf{V}_K) in each case can be obtained by Lemma 2.4 and direct computations. Let $k_{22} \neq 0$. Since $\det K = 1$,

$$\mathbf{A} = [e^{i\gamma}K] - I = \begin{bmatrix} e^{i\gamma}k_{11} & e^{i\gamma}k_{12} & -1 & 0 \\ -k_{21}/k_{22} & -1 & 0 & e^{-i\gamma}/k_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & -1 & 0 & \bar{z} \\ z & 0 & -1 & b_{22} \end{bmatrix} \in \mathcal{O}_{2,3}^{\mathbb{C}},$$

where $a_{11} := -k_{21}/k_{22}$, $z := e^{i\gamma}/k_{22}$, $b_{22} := k_{12}/k_{22}$. Then by (iii) of Lemma 2.7, $\mathbf{A} \in \mathcal{C}_{2,3,z,b_{22}} \cap \mathcal{C}_{2,3,z,a_{11}}$, and the corresponding LBCs satisfy that

$$\begin{aligned} \mathbf{S}_5 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & b_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -k_{22} & k_{12} \end{bmatrix} = \mathbf{S}_K, \\ \mathbf{S}_6 &= \begin{bmatrix} a_{11} & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} k_{21} & k_{22} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \mathbf{V}_K. \end{aligned}$$

Note that $b_{22} = 0$ is equivalent to $k_{12} = 0$; in the case that $b_{22} \neq 0$, one gets that $a_{11} + f_0 < |z|^2/b_{22}$, $a_{11} + f_0 > |z|^2/b_{22}$, and $a_{11} + f_0 = |z|^2/b_{22}$ are equivalent to $(k_{11} - f_0k_{12})k_{12} > 0$, $(k_{11} - f_0k_{12})k_{12} < 0$, and $k_{11} - f_0k_{12} = 0$, respectively; $a_{11} + f_0 = 0$ is equivalent to $f_0k_{22} - k_{21} = 0$; in the case that $a_{11} + f_0 \neq 0$, one gets that $b_{22} < |z|^2/(a_{11} + f_0)$, $b_{22} > |z|^2/(a_{11} + f_0)$, and $b_{22} = |z|^2/(a_{11} + f_0)$ are equivalent to $(k_{11} - f_0k_{12})(f_0k_{22} - k_{21}) > 0$, $(k_{11} - f_0k_{12})(f_0k_{22} - k_{21}) < 0$, and $k_{11} - f_0k_{12} = 0$, respectively. Therefore, by Theorem 3.4, one gets that $k_{12} = 0$ implies (3.20); $(k_{11} - f_0k_{12})k_{12} > 0$ implies (3.21); $(k_{11} - f_0k_{12})k_{12} < 0$ implies (3.22); $k_{11} - f_0k_{12} = 0$ implies (3.23) and (3.27); $f_0k_{22} - k_{21} = 0$ implies (3.24); $(k_{11} - f_0k_{12})(f_0k_{22} - k_{21}) > 0$ implies (3.25); $(k_{11} - f_0k_{12})(f_0k_{22} - k_{21}) < 0$ implies (3.26).

Let $k_{22} = 0$. Now we show that (3.21)–(3.23) and (3.25)–(3.27) hold in this case. Since $k_{22} = 0$, $-k_{12}k_{21} = 1$. Denote

$$K_\epsilon = \begin{pmatrix} k_{11} & k_{12} \\ (-1 + \epsilon k_{11})/k_{12} & \epsilon \end{pmatrix} \in SL(2, \mathbb{R}), \quad \epsilon \in \mathbb{R}. \quad (3.28)$$

Then $\lim_{\epsilon \rightarrow 0} K_\epsilon = K$. By the definition of \mathbf{S}_K and \mathbf{V}_K , one has that

$$\mathbf{S}_{K_\epsilon} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -\epsilon & k_{12} \end{bmatrix} \quad \text{and} \quad \mathbf{V}_{K_\epsilon} = \begin{bmatrix} -1 + \epsilon k_{11} & \epsilon k_{12} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then

$$[e^{i\gamma}K_\epsilon] - I \rightarrow [e^{i\gamma}K] - I, \quad \mathbf{S}_{K_\epsilon} \rightarrow \mathbf{S}_K, \quad \mathbf{V}_{K_\epsilon} \rightarrow \mathbf{V}_K, \quad \text{as } \epsilon \rightarrow 0.$$

In the case that $(k_{11} - f_0k_{12})k_{12} > 0$, by (3.21) for $[e^{i\gamma}K_\epsilon] - I$, $\epsilon > 0$, one gets that

$$\begin{aligned} \lambda_0(\mathbf{S}_{K_\epsilon}) &\leq \lambda_0(e^{i\gamma}K_\epsilon) \leq \lambda_1(\mathbf{S}_{K_\epsilon}) \leq \lambda_1(e^{i\gamma}K_\epsilon) \\ &\leq \cdots \leq \lambda_{N-1}(\mathbf{S}_{K_\epsilon}) \leq \lambda_{N-1}(e^{i\gamma}K_\epsilon). \end{aligned} \quad (3.29)$$

Since $k_{11} - f_0k_{12} \neq 0$ and $k_{12} \neq 0$, there are exactly N eigenvalues for each $[e^{i\gamma}K_\epsilon] - I$ and for each \mathbf{S}_{K_ϵ} , where $0 \leq \epsilon \leq 1$, by Lemma 2.4. It follows from Lemma 2.5 that $\lambda_n(e^{i\gamma}K_\epsilon)$ and $\lambda_n(\mathbf{S}_{K_\epsilon})$ are continuous in $\epsilon \in [0, 1]$, which implies that,

$$\lambda_n(e^{i\gamma}K_\epsilon) \rightarrow \lambda_n(e^{i\gamma}K), \quad \lambda_n(\mathbf{S}_{K_\epsilon}) \rightarrow \lambda_n(\mathbf{S}_K), \quad \text{as } \epsilon \rightarrow 0^+, \quad 0 \leq n \leq N-1. \quad (3.30)$$

Let $\epsilon \rightarrow 0^+$ in (3.29), it follows from (3.30) that (3.21) holds for $[e^{i\gamma}K] - I$.

With similar arguments to the proof of (3.21) for $[e^{i\gamma}K] - I$, one can show that (3.22)–(3.23) hold for $[e^{i\gamma}K] - I$.

Next, we show that (3.25)–(3.27) hold for $[e^{i\gamma}K] - I$. In the case that $-(k_{11} - f_0k_{12})k_{21} > 0$, one can choose an $\epsilon_1 > 0$ sufficiently small that $(k_{11} - f_0k_{12})(f_0\epsilon - (-1 + \epsilon k_{11})/k_{12}) = -(k_{11} - f_0k_{12})(k_{21} + \epsilon(k_{11} - f_0k_{12})/k_{12}) > 0$ and $1 - \epsilon(k_{11} - f_0k_{12}) > 0$, $0 \leq \epsilon \leq \epsilon_1$. Then by (3.25) for $[e^{i\gamma}K_\epsilon] - I$, where $0 < \epsilon \leq \epsilon_1$, one has that

$$\begin{aligned} \lambda_0(\mathbf{V}_{K_\epsilon}) &\leq \lambda_0(e^{i\gamma}K_\epsilon) \leq \lambda_1(\mathbf{V}_{K_\epsilon}) \leq \lambda_1(e^{i\gamma}K_\epsilon) \\ &\leq \cdots \leq \lambda_{N-1}(\mathbf{V}_{K_\epsilon}) \leq \lambda_{N-1}(e^{i\gamma}K_\epsilon). \end{aligned} \quad (3.31)$$

Since $k_{11} - f_0k_{12} \neq 0$ and $1 - k_{11}\epsilon + f_0k_{12}\epsilon = 1 - \epsilon(k_{11} - f_0k_{12}) > 0$, by Lemma 2.4 there are exactly N eigenvalues for each $[e^{i\gamma}K_\epsilon] - I$ and each \mathbf{V}_{K_ϵ} , where $0 \leq \epsilon \leq \epsilon_1$. By Lemma 2.5, $\lambda_n(e^{i\gamma}K_\epsilon)$ and $\lambda_n(\mathbf{V}_{K_\epsilon})$ are continuous in $\epsilon \in [0, \epsilon_1]$, which implies that

$$\lambda_n(e^{i\gamma}K_\epsilon) \rightarrow \lambda_n(e^{i\gamma}K), \quad \lambda_n(\mathbf{V}_{K_\epsilon}) \rightarrow \lambda_n(\mathbf{V}_K), \quad \text{as } \epsilon \rightarrow 0^+, \quad 0 \leq n \leq N-1. \quad (3.32)$$

Let $\epsilon \rightarrow 0^+$ in (3.31), it follows from (3.32) that (3.25) holds for $[e^{i\gamma}K] - I$.

With a similar argument to the proof of (3.25) for $[e^{i\gamma}K] - I$, one can show that (3.26)–(3.27) hold for $[e^{i\gamma}K] - I$. This completes the proof.

Remark 3.2. (ii) of Theorem 3.6 and (i) of Theorem 3.7 can also be obtained by dividing the discussion into two cases: $k_{12} \neq 0$ and $k_{12} = 0$, applying Theorem 3.3, and using a similar method to that used in the proof of them; while (i) of Theorem 3.6 and (ii) of Theorem 3.7 can also be obtained by dividing the discussion into two cases: $k_{21} \neq 0$ and $k_{21} = 0$, applying Theorem 3.5, and using a similar method to that used in the proof of them.

The following result, which is a direct consequence of Theorems 3.6–3.7, gives comparison of eigenvalues for $[e^{i\gamma}K] - I$ with those for \mathbf{S}_K , those for \mathbf{U}_K , those for \mathbf{T}_K , and those for \mathbf{V}_K under the assumption that $k_{11} - f_0k_{12} = 0$.

Corollary 3.2. Fix a difference equation $\omega = (1/f, q, w)$. Let $\mathbf{A} = [e^{i\gamma}K] - I \in \mathcal{B}^\mathbb{C}$, where $K \in SL(2, \mathbb{R})$ and $\gamma \in (-\pi, \pi]$. If $k_{11} - f_0k_{12} = 0$, then

$$\begin{aligned} \{\lambda_0(\mathbf{S}_K), \lambda_0(\mathbf{T}_K), \lambda_0(\mathbf{V}_K)\} &\leq \lambda_0(e^{i\gamma}K) \leq \{\lambda_1(\mathbf{S}_K), \lambda_1(\mathbf{T}_K), \lambda_1(\mathbf{V}_K), \lambda_0(\mathbf{U}_K)\} \leq \\ \lambda_1(e^{i\gamma}K) &\leq \{\lambda_2(\mathbf{S}_K), \lambda_2(\mathbf{T}_K), \lambda_2(\mathbf{V}_K), \lambda_1(\mathbf{U}_K)\} \leq \cdots \leq \lambda_{N-3}(e^{i\gamma}K) \leq \{\lambda_{N-2}(\mathbf{S}_K), \\ \lambda_{N-2}(\mathbf{T}_K), \lambda_{N-2}(\mathbf{V}_K), \lambda_{N-3}(\mathbf{U}_K)\} &\leq \lambda_{N-2}(e^{i\gamma}K) \leq \{\lambda_{N-1}(\mathbf{S}_K), \lambda_{N-1}(\mathbf{T}_K), \lambda_{N-1}(\mathbf{V}_K)\}. \end{aligned}$$

Note that a coupled BC $[e^{i\gamma}K] - I$ can be written as $[e^{i\gamma/2}K] - e^{-i\gamma/2}I$. Then by Lemma 2.3, a simple calculation yields that

$$\Gamma(\lambda) = 2 \cos \gamma - k_{22}\phi_N(\lambda) + k_{21}\psi_N(\lambda) + k_{12}f_N\Delta\phi_N(\lambda) - k_{11}f_N\Delta\psi_N(\lambda). \quad (3.33)$$

Thus, the eigenvalues for $[e^{i\gamma}K| - I]$ are the same as those for $[e^{-i\gamma}K| - I]$ by (3.33). Now, it's ready to establish inequalities among eigenvalues for the three coupled BCs: $[K| - I]$, $[e^{i\gamma}K| - I]$, and $[-K| - I]$, and those for the corresponding separated ones.

Theorem 3.8. *Fix a difference equation $\omega = (1/f, q, w)$ satisfying that $\prod_{i=0}^{N-1}(1/f_i) > 0$. Let $\gamma \in (-\pi, 0) \cup (0, \pi)$ and $K \in SL(2, \mathbb{R})$ satisfy that $k_{11} - f_0 k_{12} \neq 0$. Then the eigenvalues of SLPs $(\omega, [K| - I])$, $(\omega, [e^{i\gamma}K| - I])$, $(\omega, [-K| - I])$, and (ω, \mathbf{S}_K) satisfy the following inequalities:*

(i) *for $k_{11} - f_0 k_{12} > 0$ and $k_{12} > 0$,*

$$\begin{aligned} \lambda_0(\mathbf{S}_K) &\leq \lambda_0(K) < \lambda_0(e^{i\gamma}K) < \lambda_0(-K) \leq \lambda_1(\mathbf{S}_K) \leq \lambda_1(-K) < \\ \lambda_1(e^{i\gamma}K) &< \lambda_1(K) \leq \cdots \leq \lambda_{N-2}(\mathbf{S}_K) \leq \lambda_{N-2}(K) < \lambda_{N-2}(e^{i\gamma}K) < \\ \lambda_{N-2}(-K) &\leq \lambda_{N-1}(\mathbf{S}_K) \leq \lambda_{N-1}(-K) < \lambda_{N-1}(e^{i\gamma}K) < \lambda_{N-1}(K) \end{aligned} \quad (3.34)$$

in the case that N is even;

$$\begin{aligned} \lambda_0(\mathbf{S}_K) &\leq \lambda_0(K) < \lambda_0(e^{i\gamma}K) < \lambda_0(-K) \leq \lambda_1(\mathbf{S}_K) \leq \lambda_1(-K) < \\ \lambda_1(e^{i\gamma}K) &< \lambda_1(K) \leq \cdots \leq \lambda_{N-2}(\mathbf{S}_K) \leq \lambda_{N-2}(-K) < \lambda_{N-2}(e^{i\gamma}K) < \\ \lambda_{N-2}(K) &\leq \lambda_{N-1}(\mathbf{S}_K) \leq \lambda_{N-1}(K) < \lambda_{N-1}(e^{i\gamma}K) < \lambda_{N-1}(-K) \end{aligned} \quad (3.35)$$

in the case that N is odd;

(ii) *for $k_{11} - f_0 k_{12} > 0$ and $k_{12} < 0$,*

$$\begin{aligned} \lambda_0(K) &< \lambda_0(e^{i\gamma}K) < \lambda_0(-K) \leq \lambda_0(\mathbf{S}_K) \leq \lambda_1(-K) < \lambda_1(e^{i\gamma}K) < \lambda_1(K) \\ &\leq \lambda_1(\mathbf{S}_K) \leq \cdots \leq \lambda_{N-2}(K) < \lambda_{N-2}(e^{i\gamma}K) < \lambda_{N-2}(-K) \leq \\ \lambda_{N-2}(\mathbf{S}_K) &\leq \lambda_{N-1}(-K) < \lambda_{N-1}(e^{i\gamma}K) < \lambda_{N-1}(K) \leq \lambda_{N-1}(\mathbf{S}_K) \end{aligned}$$

in the case that N is even;

$$\begin{aligned} \lambda_0(K) &< \lambda_0(e^{i\gamma}K) < \lambda_0(-K) \leq \lambda_0(\mathbf{S}_K) \leq \lambda_1(-K) < \lambda_1(e^{i\gamma}K) < \lambda_1(K) \\ &\leq \lambda_1(\mathbf{S}_K) \leq \cdots \leq \lambda_{N-2}(-K) < \lambda_{N-2}(e^{i\gamma}K) < \lambda_{N-2}(K) \leq \\ \lambda_{N-2}(\mathbf{S}_K) &\leq \lambda_{N-1}(K) < \lambda_{N-1}(e^{i\gamma}K) < \lambda_{N-1}(-K) \leq \lambda_{N-1}(\mathbf{S}_K) \end{aligned}$$

in the case that N is odd;

(iii) *for $k_{11} > 0$ and $k_{12} = 0$,*

$$\begin{aligned} \lambda_0(K) &< \lambda_0(e^{i\gamma}K) < \lambda_0(-K) \leq \lambda_0(\mathbf{S}_K) \leq \lambda_1(-K) < \lambda_1(e^{i\gamma}K) \\ &< \lambda_1(K) \leq \lambda_1(\mathbf{S}_K) \leq \cdots \leq \lambda_{N-2}(K) < \lambda_{N-2}(e^{i\gamma}K) < \lambda_{N-2}(-K) \\ &\leq \lambda_{N-2}(\mathbf{S}_K) \leq \lambda_{N-1}(-K) < \lambda_{N-1}(e^{i\gamma}K) < \lambda_{N-1}(K) \end{aligned}$$

in the case that N is even;

$$\begin{aligned}
& \lambda_0(K) < \lambda_0(e^{i\gamma}K) < \lambda_0(-K) \leq \lambda_0(\mathbf{S}_K) \leq \lambda_1(-K) < \lambda_1(e^{i\gamma}K) \\
& < \lambda_1(K) \leq \lambda_1(\mathbf{S}_K) \leq \cdots \leq \lambda_{N-2}(-K) < \lambda_{N-2}(e^{i\gamma}K) < \lambda_{N-2}(K) \\
& \leq \lambda_{N-2}(\mathbf{S}_K) \leq \lambda_{N-1}(K) < \lambda_{N-1}(e^{i\gamma}K) < \lambda_{N-1}(-K)
\end{aligned}$$

in the case that N is odd;

(iv) assertions in (i)–(iii) hold with K replaced by $-K$.

Proof. First, we show that (i) holds. We only show that (3.34) holds, since (3.35) can be shown similarly. By (2.3) and (3.33), one can easily verify that the leading term of $\Gamma(\lambda)$ as a polynomial of λ is

$$(-1)^{N+1} 1/f_0(w_N \prod_{i=1}^{N-1} (w_i/f_i))(k_{11} - f_0 k_{12}) \lambda^N.$$

Since $k_{11} - f_0 k_{12} > 0$ and $1/f_0(w_N \prod_{i=1}^{N-1} (w_i/f_i)) > 0$, one has that

$$\lim_{\lambda \rightarrow -\infty} \Gamma(\lambda) = -\infty, \quad \lim_{\lambda \rightarrow +\infty} \Gamma(\lambda) = -\infty, \quad (3.36)$$

in the case that N is even.

Let $\gamma \in (-\pi, 0) \cup (0, \pi)$. By Lemma 2.3, $\lambda_n(e^{i\gamma}K)$, $0 \leq n \leq N-1$, are exactly the zeros of the polynomial $\Gamma(\lambda)$ for $(\omega, [e^{i\gamma}K] - I)$. It follows from Theorem 3.1 in [23] that $\lambda_n(e^{i\gamma}K)$ is a simple eigenvalue for each $0 \leq n \leq N-1$. Thus by Rolle mean value Theorem, there are exactly $N-1$ real zeros for $\Gamma'(\lambda)$ and they are denoted by x_1, \dots, x_{N-1} . Then $x_n \in (\lambda_{n-1}(e^{i\gamma}K), \lambda_n(e^{i\gamma}K))$, and $\Gamma(\lambda)$ is strictly increasing in $(-\infty, x_1)$ and strictly decreasing in (x_1, x_2) . Hence $\Gamma(x_1) > 0$. From (3.33), (3.36), and the above discussion, it follows that $\lambda_0(K) < \lambda_0(e^{i\gamma}K) < \lambda_0(-K)$ and $\lambda_1(-K) < \lambda_1(e^{i\gamma}K) < \lambda_1(K)$. Similarly, one can show that

$$\begin{aligned}
& \lambda_j(K) < \lambda_j(e^{i\gamma}K) < \lambda_j(-K), \quad j = 0, 2, 4, \dots, N-2, \\
& \lambda_j(-K) < \lambda_j(e^{i\gamma}K) < \lambda_j(K), \quad j = 1, 3, 5, \dots, N-1.
\end{aligned} \quad (3.37)$$

Since $(k_{11} - f_0 k_{12})k_{12} > 0$, it follows from (3.21) that

$$\begin{aligned}
& \lambda_0(\mathbf{S}_K) \leq \{\lambda_0(e^{i\gamma}K) : \gamma \in (-\pi, \pi]\} \leq \lambda_1(\mathbf{S}_K) \leq \{\lambda_1(e^{i\gamma}K) : \gamma \in (-\pi, \pi]\} \\
& \leq \cdots \leq \lambda_{N-1}(\mathbf{S}_K) \leq \{\lambda_{N-1}(e^{i\gamma}K) : \gamma \in (-\pi, \pi]\}.
\end{aligned} \quad (3.38)$$

Therefore, (3.37)–(3.38) imply that (3.34) holds.

Assertions (ii)–(iii) can be shown similarly.

Now we show that (iv) holds. It follows from the definition of \mathbf{S}_K in Theorem 3.7 that $\mathbf{S}_K = \mathbf{S}_{-K}$. Since $k_{11} - f_0 k_{12} \neq 0$ and the entries of K satisfy none of the conditions in (i), (ii) or (iii), there are exactly three cases:

(i') $k_{11} - f_0 k_{12} < 0$, $k_{12} < 0$;

(ii') $k_{11} - f_0 k_{12} < 0$, $k_{12} > 0$;

(iii') $k_{11} < 0$, $k_{12} = 0$.

If the entries of K satisfy (i'), (ii'), and (iii'), separately, then assertions in (i), (ii), and (iii) hold for $-K$, respectively. This completes the proof.

The following result is a direct consequence of Theorems 3.6 and 3.8.

Corollary 3.3. *Fix a difference equation $\omega = (1/f, q, w)$ satisfying that $\prod_{i=0}^{N-1} (1/f_i) > 0$. Let $K \in SL(2, \mathbb{R})$.*

- (i) *If $k_{11} - f_0 k_{12} > 0$, then there are exactly N eigenvalues for $[K] - I$ and $[-K] - I$, separately. Further, whether N is odd or even, $\lambda_0(K)$ is a simple eigenvalue; if $\lambda_j(K) < \lambda_{j+1}(K)$ for some odd number j ($1 \leq j \leq N-2$), then $\lambda_j(K)$ and $\lambda_{j+1}(K)$ are simple eigenvalues. Similar results hold in the case that $\lambda_j(-K) < \lambda_{j+1}(-K)$ for some even number j ($0 \leq j \leq N-2$). If N is odd, then $\lambda_{N-1}(-K)$ is a simple eigenvalue; and if N is even, then $\lambda_{N-1}(K)$ is a simple eigenvalue.*
- (ii) *If $k_{11} - f_0 k_{12} < 0$, similar results in (i) can be obtained with K replaced by $-K$.*

Remark 3.3. Theorem 3.1 in [18] gives inequalities among eigenvalues for $[K] - I$, those for $[e^{i\gamma}K] - I$, and those for $[-K] - I$ in the case that $k_{12} = 0$ under the assumption that $f_0 = f_N = 1$. They are direct consequences of (iii)–(iv) in Theorem 3.8.

With the help of Theorems 3.6–3.7, (3.33), and a similar method to that used in the proof of Theorem 3.8, one can deduce the following Theorems 3.9–3.11:

Theorem 3.9. *Fix a difference equation $\omega = (1/f, q, w)$ satisfying that $\prod_{i=0}^{N-1} (1/f_i) > 0$. Let $\gamma \in (-\pi, 0) \cup (0, \pi)$ and $K \in SL(2, \mathbb{R})$ satisfy that $k_{11} - f_0 k_{12} \neq 0$. Then the eigenvalues of SLPs $(\omega, [K] - I)$, $(\omega, [e^{i\gamma}K] - I)$, $(\omega, [-K] - I)$, and (ω, \mathbf{U}_K) satisfy the following inequalities:*

- (i) *if $k_{11} - f_0 k_{12} > 0$, then*

$$\begin{aligned} \lambda_0(K) &< \lambda_0(e^{i\gamma}K) < \lambda_0(-K) \leq \lambda_0(\mathbf{U}_K) \leq \lambda_1(-K) < \\ \lambda_1(e^{i\gamma}K) &< \lambda_1(K) \leq \lambda_1(\mathbf{U}_K) \leq \cdots \leq \lambda_{N-2}(K) < \lambda_{N-2}(e^{i\gamma}K) < \\ \lambda_{N-2}(-K) &\leq \lambda_{N-2}(\mathbf{U}_K) \leq \lambda_{N-1}(-K) < \lambda_{N-1}(e^{i\gamma}K) < \lambda_{N-1}(K) \end{aligned}$$

in the case that N is even;

$$\begin{aligned} \lambda_0(K) &< \lambda_0(e^{i\gamma}K) < \lambda_0(-K) \leq \lambda_0(\mathbf{U}_K) \leq \lambda_1(-K) < \\ \lambda_1(e^{i\gamma}K) &< \lambda_1(K) \leq \lambda_1(\mathbf{U}_K) \leq \cdots \leq \lambda_{N-2}(-K) < \lambda_{N-2}(e^{i\gamma}K) < \\ \lambda_{N-2}(K) &\leq \lambda_{N-2}(\mathbf{U}_K) \leq \lambda_{N-1}(K) < \lambda_{N-1}(e^{i\gamma}K) < \lambda_{N-1}(-K) \end{aligned}$$

in the case that N is odd;

- (ii) *assertions in (i) hold with K replaced by $-K$.*

Theorem 3.10. Fix a difference equation $\omega = (1/f, q, w)$ satisfying that $\prod_{i=0}^{N-1}(1/f_i) > 0$. Let $\gamma \in (-\pi, 0) \cup (0, \pi)$ and $K \in SL(2, \mathbb{R})$ satisfy that $k_{11} - f_0 k_{12} \neq 0$. Then (i)–(ii) in Theorem 3.8 hold with $k_{12} > 0$, $k_{12} < 0$, and $\lambda_n(\mathbf{S}_K)$ replaced by $f_0 k_{11} > 0$, $f_0 k_{11} < 0$, and $\lambda_n(\mathbf{T}_K)$, respectively, where $0 \leq n \leq N-1$; (iii) in Theorem 3.8 holds with $k_{11} > 0$, $k_{12} = 0$, and $\lambda_n(\mathbf{S}_K)$ replaced by $f_0 k_{12} < 0$, $k_{11} = 0$, and $\lambda_n(\mathbf{T}_K)$, respectively, where $0 \leq n \leq N-2$; (iv) in Theorem 3.8 also holds.

Theorem 3.11. Fix a difference equation $\omega = (1/f, q, w)$ satisfying that $\prod_{i=0}^{N-1}(1/f_i) > 0$. Let $\gamma \in (-\pi, 0) \cup (0, \pi)$ and $K \in SL(2, \mathbb{R})$ satisfy that $k_{11} - f_0 k_{12} \neq 0$. Then (i)–(ii) in Theorem 3.8 hold with $k_{12} > 0$, $k_{12} < 0$, and $\lambda_n(\mathbf{S}_K)$ replaced by $f_0 k_{22} - k_{21} > 0$, $f_0 k_{22} - k_{21} < 0$, and $\lambda_n(\mathbf{V}_K)$, respectively, where $0 \leq n \leq N-1$; (iii) in Theorem 3.8 holds with $k_{11} > 0$, $k_{12} = 0$, and $\lambda_n(\mathbf{S}_K)$ replaced by $k_{11} - f_0 k_{12} > 0$, $f_0 k_{22} - k_{21} = 0$, and $\lambda_n(\mathbf{V}_K)$, respectively, where $0 \leq n \leq N-2$; (iv) in Theorem 3.8 also holds.

Remark 3.4. We have not given the similar inequalities as those in Theorems 3.8–3.11 in the case that $k_{11} - f_0 k_{12} = 0$ since it is not clear that what the limits of the polynomial $\Gamma(\lambda)$ given in (3.33) is as $\lambda \rightarrow \pm\infty$ in this case.

3.4. Inequalities among eigenvalues for different coupled BCs

In this subsection, we shall establish inequalities among eigenvalues for different coupled BCs applying Theorems 3.2–3.3.

For each $K \in SL(2, \mathbb{R})$, we set

$$\widehat{K} := \begin{pmatrix} k_{11} & k_{11}/f_0 \\ k_{21} & (f_0 + k_{11}k_{21})/(k_{11}f_0) \end{pmatrix} \quad \text{if } k_{11} \neq 0; \quad (3.39)$$

and

$$\widetilde{K} := \begin{pmatrix} f_0 k_{12} & k_{12} \\ (f_0 k_{12} k_{22} - 1)/k_{12} & k_{22} \end{pmatrix} \quad \text{if } k_{12} \neq 0. \quad (3.40)$$

Note that $\widehat{K}, \widetilde{K} \in SL(2, \mathbb{R})$, and $K = \widehat{K} = \widetilde{K}$ if $k_{11} - f_0 k_{12} = 0$. The next result compares eigenvalues for $[e^{i\gamma}K] - I$ with those for $[e^{i\gamma}\widehat{K}] - I$, and eigenvalues for $[e^{i\gamma}K] - I$ with those for $[e^{i\gamma}\widetilde{K}] - I$, separately.

Theorem 3.12. Fix a difference equation $\omega = (1/f, q, w)$. Let $[e^{i\gamma}K] - I \in \mathcal{B}^{\mathbb{C}}$, where $\gamma \in (-\pi, \pi]$ and $K \in SL(2, \mathbb{R})$ satisfies that $k_{11} - f_0 k_{12} \neq 0$. Then there are exactly N eigenvalues for $[e^{i\gamma}K] - I$, and exactly $N-1$ eigenvalues for both $[e^{i\gamma}\widehat{K}] - I$ and $[e^{i\gamma}\widetilde{K}] - I$, where \widehat{K} and \widetilde{K} are defined by (3.39)–(3.40). Furthermore, we have that

(i) if $k_{11} \neq 0$, then

$$\begin{aligned} \lambda_0(e^{i\gamma}K) &\leq \lambda_0(e^{i\gamma}\widehat{K}) \leq \lambda_1(e^{i\gamma}K) \leq \lambda_1(e^{i\gamma}\widehat{K}) \leq \cdots \leq \\ \lambda_{N-2}(e^{i\gamma}K) &\leq \lambda_{N-2}(e^{i\gamma}\widetilde{K}) \leq \lambda_{N-1}(e^{i\gamma}K); \end{aligned}$$

(ii) if $k_{12} \neq 0$, then

$$\begin{aligned}\lambda_0(e^{i\gamma}K) &\leq \lambda_0(e^{i\gamma}\tilde{K}) \leq \lambda_1(e^{i\gamma}K) \leq \lambda_1(e^{i\gamma}\tilde{K}) \leq \cdots \leq \\ \lambda_{N-2}(e^{i\gamma}K) &\leq \lambda_{N-2}(e^{i\gamma}\tilde{K}) \leq \lambda_{N-1}(e^{i\gamma}K).\end{aligned}$$

Proof. By Lemma 2.4, the number of eigenvalues for each BC can be obtained directly. Firstly, we show that (i) holds. Let $a_{12} := k_{12}/k_{11}$, $z := -e^{i\gamma}/k_{11}$, and $b_{21} := -k_{21}/k_{11}$. $\mathbf{A}(s)$ has the same meaning as that in Lemma 2.7. Then a direct computation implies that $[e^{i\gamma}K| - I] = \mathbf{A}(a_{12})$ and $[e^{i\gamma}\tilde{K}| - I] = \mathbf{A}(1/f_0)$. Hence, (i) holds by Theorem 3.2.

Assertion (ii) can be shown similarly to that for (i) by Theorem 3.3. The proof is complete.

Remark 3.5. The inequalities in Theorem 3.12 may not be strict. See Example 3.1.

4. Inequalities among eigenvalues for different equations

In this section, inequalities among eigenvalues for equations with different coefficients and weight functions are established by applying the monotonicity result of λ_n in Theorems 3.1–3.3 in [22].

Fix a self-adjoint BC

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \end{bmatrix}$$

in this section. Let $\mu_1 := a_{11}b_{22} - a_{21}b_{12}$, $\mu_2 := a_{22}b_{12} - a_{12}b_{22}$, and $\eta := -\mu_2/\mu_1$ if $\mu_1 \neq 0$. If $\mu_1 = 0$ and $\mu_2 = 0$, then the BC \mathbf{A} can be written as

$$\text{either } \mathbf{A}_1 := \begin{bmatrix} a_{11} & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \text{ or } \mathbf{A}_2 := \begin{bmatrix} 1 & a_{12} & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}. \quad (4.1)$$

Firstly, we give two lemmas, which play important roles in establishing inequalities among eigenvalues for equations with different weight functions. Fix $f = \{1/f_n\}_{n=0}^N$ and $q = \{q_n\}_{n=1}^N$. By Lemma 2.4, the number of eigenvalues of $((1/f, q, w), \mathbf{A})$ is independent of w . Thus, we assume that $((1/f, q, w), \mathbf{A})$ has exactly k ($N-2 \leq k \leq N$) eigenvalues for each $w \in \mathbb{R}^{N,+}$ in the following two lemmas:

Lemma 4.1. Fix $f = \{f_n\}_{n=0}^N$, $q = \{q_n\}_{n=1}^N$, $1 \leq i \leq N$, $w_1^{(0)}, \dots, w_{i-1}^{(0)}, w_{i+1}^{(0)}, \dots, w_N^{(0)}$, a self-adjoint BC \mathbf{A} , and $1 \leq j \leq k$. Let $\lambda_j(w_i) := \lambda_j(w_1^{(0)}, \dots, w_{i-1}^{(0)}, w_i, w_{i+1}^{(0)}, \dots, w_N^{(0)}, \mathbf{A})$ be the j -th eigenvalue function in the w_i -direction. If $\lambda_j(w_i^{(0)}) = 0$ for some $w_i^{(0)} \in \mathbb{R}^+$, then $\lambda_j(w_i) = 0$ for all $w_i > w_i^{(0)}$.

Proof. Suppose that there exists $w'_i > w_i^{(0)}$ such that $\lambda_j(w'_i) \neq 0$. By Theorems 3.1–3.3 in [22], $\lambda_j(w_i)$ is continuous in $w_i \in \mathbb{R}^+$, and its positive and negative parts are non-increasing and non-decreasing in w_i -direction, respectively. This is a contradiction to that $\lambda_j(w'_i) \neq 0$. The proof is complete.

Lemma 4.2. Fix $f = \{f_n\}_{n=0}^N$, $q = \{q_n\}_{n=1}^N$, a self-adjoint BC \mathbf{A} , and $1 \leq j \leq k$. Let $\lambda_j(w) := \lambda_j(1/f, q, w, \mathbf{A})$ be the j -th eigenvalue function for $w \in \mathbb{R}^{N,+}$. Then either $\lambda_j(w) \geq 0$ for all $w \in \mathbb{R}^{N,+}$ or $\lambda_j(w) \leq 0$ for all $w \in \mathbb{R}^{N,+}$.

Proof. In the case that there exists $w^{(0)} \in \mathbb{R}^{N,+}$ such that $\lambda_j(w^{(0)}) > 0$, we shall show that $\lambda_j(w_i) \geq 0$ for each $1 \leq i \leq N$ and all $w_i \in \mathbb{R}^+$, where $\lambda_j(w_i)$ is defined in Lemma 4.1. Otherwise, there exists a $w'_i \in \mathbb{R}^+$ such that $\lambda_j(w'_i) < 0$. Without loss of generality, assume that $w'_i < w_i^{(0)}$. Then there must exist $w''_i \in (w'_i, w_i^{(0)})$ such that $\lambda_j(w''_i) = 0$ by the continuity of $\lambda_j(w_i)$ in $w_i \in \mathbb{R}^+$. By Lemma 4.1, $\lambda_j(w_i) = 0$ for all $w_i > w''_i$. This is contradict to $\lambda_j(w_i^{(0)}) > 0$. Thus $\lambda_j(w_i) \geq 0$ for all $w_i \in \mathbb{R}^+$. This, together with the monotonicity of λ_j in each w_i -direction, $1 \leq i \leq N$, implies that $\lambda_j(w) \geq 0$ for all $w \in \mathbb{R}^{N,+}$.

In the case that there exists $w^{(0)} \in \mathbb{R}^{N,+}$ such that $\lambda_j(w^{(0)}) < 0$, with a similar argument above, one can show that $\lambda_j(w) \leq 0$ for all $w \in \mathbb{R}^{N,+}$.

If it is not one of the above two cases, then $\lambda_j(w) \equiv 0$ for all $w \in \mathbb{R}^{N,+}$. The proof is complete.

Now, inequalities among eigenvalues for equations with different coefficients and weight functions are established.

Theorem 4.1. Fix a self-adjoint BC \mathbf{A} . Consider the following two different equations:

$$-\nabla(f_n^{(i)} \Delta y_n) + q_n^{(i)} y_n = \lambda w_n^{(i)} y_n, \quad n \in [1, N], \quad i = 1, 2, \quad (4.2)_i$$

and the same BC \mathbf{A} . By $\lambda_n^{(i)}$ denote the n -th eigenvalue of $(4.2)_i$ and \mathbf{A} . Let $f_j^{(1)} \leq f_j^{(2)}$ for $0 \leq j \leq N-1$, $q_m^{(1)} \leq q_m^{(2)}$ for $1 \leq m \leq N$, and $f_N^{(1)}$ and $f_N^{(2)}$ be two given non-zero real numbers.

(i) If one of the following conditions (1)–(2) holds,

(1) $\mu_1 \neq 0$, $\mu_2 \neq 0$, and either $f_0^{(2)} \in (-\infty, 1/\eta)$ or $f_0^{(1)} \in (1/\eta, +\infty)$;

(2) either $\mu_1 = 0$, $\mu_2 \neq 0$ or $\mu_1 \neq 0$, $\mu_2 = 0$;

then there are exactly N eigenvalues $\lambda_n^{(i)}$ of $(4.2)_i$ and \mathbf{A} , where $i = 1, 2$. Further, for any given $0 \leq n \leq N-1$,

(a) if $\lambda_n^{(1)} > 0$ and $w_m^{(1)} \geq w_m^{(2)}$, $1 \leq m \leq N$, then

$$\lambda_n^{(1)} \leq \lambda_n^{(2)}; \quad (4.3)$$

(b) if $\lambda_n^{(1)} \leq 0$ and $w_m^{(1)} \leq w_m^{(2)}$, $1 \leq m \leq N$, then (4.3) holds.

(ii) If one of the following conditions (3)–(7) holds,

(3) $\mu_1 \neq 0$, $\mu_2 \neq 0$, and $f_0^{(1)} = f_0^{(2)} = 1/\eta$;

(4) $\mu_1 = 0, \mu_2 = 0, \mathbf{A} = \mathbf{A}_1$ with $a_{11} \neq 0$, and either $f_0^{(2)} \in (-\infty, -a_{11})$ or $f_0^{(1)} \in (-a_{11}, +\infty)$;

(5) $\mu_1 = 0, \mu_2 = 0, \mathbf{A} = \mathbf{A}_2$ with $a_{12} \neq 0$, and either $f_0^{(2)} \in (-\infty, 1/a_{12})$ or $f_0^{(1)} \in (1/a_{12}, +\infty)$, where \mathbf{A}_1 and \mathbf{A}_2 are specified in (4.1);

(6) $\mu_1 = 0, \mu_2 = 0$, and $\mathbf{A} = \mathbf{A}_1$ with $a_{11} = 0$;

(7) $\mu_1 = 0, \mu_2 = 0$, and $\mathbf{A} = \mathbf{A}_2$ with $a_{12} = 0$;

then there are exactly $N-1$ eigenvalues $\lambda_n^{(i)}$ of $(4.2)_i$ and \mathbf{A} , where $i = 1, 2$. Further, for any given $0 \leq n \leq N-2$, assertions (a)–(b) in (i) hold.

(iii) If one of the following conditions (8)–(9) holds,

(8) $\mu_1 = 0, \mu_2 = 0, \mathbf{A} = \mathbf{A}_1$ with $a_{11} \neq 0$, and $f_0^{(1)} = f_0^{(2)} = -a_{11}$;

(9) $\mu_1 = 0, \mu_2 = 0, \mathbf{A} = \mathbf{A}_2$ with $a_{12} \neq 0$, and $f_0^{(1)} = f_0^{(2)} = 1/a_{12}$;

then there are exactly $N-2$ eigenvalues $\lambda_n^{(i)}$ of $(4.2)_i$ and \mathbf{A} , where $i = 1, 2$. Further, for any given $0 \leq n \leq N-3$, assertions (a)–(b) in (i) hold.

Proof. The number of eigenvalues of $(4.2)_i$ and \mathbf{A} in each case can be obtained by Lemma 2.4. Firstly, we show that (i) holds with the assumption (1). Let $0 \leq n \leq N-1$. In the case that $\lambda_n^{(1)} > 0$, $\lambda_n(f^{(1)}, q^{(1)}, w) \geq 0$ for all $w \in \mathbb{R}^{N,+}$ by Lemma 4.2. By Theorem 3.1 in [22], $\lambda_n(f^{(1)}, q^{(1)}, w)$ is non-increasing in each w_m -direction, $1 \leq m \leq N$. Thus, if $w_m^{(1)} \geq w_m^{(2)}$, $1 \leq m \leq N$, then

$$\lambda_n^{(1)} = \lambda_n(f^{(1)}, q^{(1)}, w^{(1)}) \leq \lambda_n(f^{(1)}, q^{(1)}, w^{(2)}). \quad (4.4)$$

Again by Theorem 3.1 in [22], $\lambda_n(f, q, w^{(2)})$ is non-decreasing in $f_j \in (-\infty, 1/\eta)$ or $(1/\eta, +\infty)$ in each f_j -direction, $0 \leq j \leq N-1$; and in $q_m \in \mathbb{R}$ in each q_m -direction, $1 \leq m \leq N$. Since $f_j^{(1)} \leq f_j^{(2)} < 1/\eta$ or $1/\eta < f_j^{(1)} \leq f_j^{(2)}$, $0 \leq j \leq N-1$, $q_m^{(1)} \leq q_m^{(2)}$, $1 \leq m \leq N$, thus

$$\lambda_n(f^{(1)}, q^{(1)}, w^{(2)}) \leq \lambda_n(f^{(2)}, q^{(2)}, w^{(2)}) = \lambda_n^{(2)}. \quad (4.5)$$

(4.4)–(4.5) imply (4.3) holds.

In the case that $\lambda_n^{(1)} \leq 0$, with a similar method above, one can show that (4.3) holds.

With a similar argument to that in the proof of (i) with the assumption (1), one can show that (i) with the assumption (2), (ii)–(iii) hold. This completes the proof.

Remark 4.1. Theorem 5.5 of [16] and Theorem 3.6 of [17] give several similar inequalities as those in Theorem 4.1 with the assumption that $f_0^{(1)} = f_0^{(2)}$ and $f_N^{(1)} = f_N^{(2)}$. In addition, it is required in Theorem 5.5 of [16] that $w^{(1)} = w^{(2)}$. Note that it is not required in Theorem 4.1 that $f_N^{(1)} = f_N^{(2)}$ and $w^{(1)} = w^{(2)}$; and it is not required in (1)–(2) and (4)–(7)

in Theorem 4.1 that $f_0^{(1)} = f_0^{(2)}$. Thus, Theorem 4.1 can be regarded as a generalization of the corresponding results in Theorem 5.5 of [16] and Theorem 3.6 of [17].

Combining Theorems 3.2 and 4.1 yields inequalities among eigenvalues of SLPs with different equations and BCs in $\mathcal{O}_{1,4}^{\mathbb{C}}$.

Corollary 4.1. Consider the following two different SLPs: $(4.2)_i$ and BCs

$$\mathbf{A}(a_{12}^{(i)}, b_{21}^{(i)}) = \begin{bmatrix} 1 & a_{12}^{(i)} & \bar{z} & 0 \\ 0 & z & b_{21}^{(i)} & 1 \end{bmatrix}, \quad i = 1, 2. \quad (4.6)_i$$

By $\lambda_n^{(i)}$ denote the n -th eigenvalue of $(4.2)_i$ and $(4.6)_i$. Let $f_j^{(1)} \leq f_j^{(2)}$, $0 \leq j \leq N-1$, $q_m^{(1)} \leq q_m^{(2)}$, $1 \leq m \leq N$, $f_N^{(1)}$ and $f_N^{(2)}$ be two given non-zero real numbers, $a_{12}^{(1)} \leq a_{12}^{(2)}$, and $b_{22}^{(1)} \leq b_{22}^{(2)}$.

(i) If one of the following two conditions (1)–(2) holds,

(1) $a_{12}^{(1)} \neq 0$, $f_0^{(2)} a_{12}^{(1)} > 0$, and either $a_{12}^{(2)} < 1/f_0^{(2)}$ or $f_0^{(1)} > 1/a_{12}^{(1)}$;

(2) $a_{12}^{(1)} = 0$, and either $a_{12}^{(2)} \leq 1/f_0^{(2)}$ or $1/f_0^{(2)} < 0$;

then there are exactly N eigenvalues $\lambda_n^{(i)}$ of $(4.2)_i$ and $(4.6)_i$, $i = 1, 2$. Further, for any given $0 \leq n \leq N-1$,

(a) if $\lambda_n^{(1)} > 0$ and $w_m^{(1)} \geq w_m^{(2)}$, $1 \leq m \leq N$, then

$$\lambda_n^{(1)} \leq \lambda_n^{(2)}; \quad (4.7)$$

(b) if $\lambda_n^{(1)} \leq 0$ and $w_m^{(1)} \leq w_m^{(2)}$, $1 \leq m \leq N$, then (4.7) holds.

(ii) If $a_{12}^{(1)} = a_{12}^{(2)} = 1/f_0^{(1)} = 1/f_0^{(2)}$, then there are exactly $N-1$ eigenvalues of $(4.2)_i$ and $(4.6)_i$, $i = 1, 2$. Further, for any given $0 \leq n \leq N-2$, assertions (a)–(b) in (i) hold.

Proof. Firstly, we show that (i) holds with the assumption (1). Direct computations imply that $\mu_1^{(i)} := a_{11}^{(i)} b_{22}^{(i)} - a_{21}^{(i)} b_{12}^{(i)} = 1$, $\mu_2^{(i)} := a_{22}^{(i)} b_{12}^{(i)} - a_{12}^{(i)} b_{21}^{(i)} = -a_{12}^{(i)} \neq 0$, and $\eta^{(i)} := -\mu_2^{(i)} / \mu_1^{(i)} = a_{12}^{(i)}$, $i = 1, 2$. If $a_{12}^{(2)} < 1/f_0^{(2)}$, then $f_0^{(2)} < 1/a_{12}^{(1)} = 1/\eta^{(1)}$ since $a_{12}^{(1)} \leq a_{12}^{(2)}$ and $f_0^{(2)} a_{12}^{(1)} > 0$. If $f_0^{(1)} > 1/a_{12}^{(1)}$, then $f_0^{(1)} > 1/\eta^{(1)}$. Fix the BC $\mathbf{A}(a_{12}^{(1)}, b_{21}^{(1)})$. By (1) of Theorem 4.1, one gets that there are exactly N eigenvalues of $(4.2)_1$ – $(4.6)_1$ and $(4.2)_2$ – $(4.6)_1$, and in either case (a) or (b), for each $0 \leq n \leq N-1$,

$$\lambda_n^{(1)} = \lambda_n(1/f^{(1)}, q^{(1)}, w^{(1)}, \mathbf{A}(a_{12}^{(1)}, b_{21}^{(1)})) \leq \lambda_n(1/f^{(2)}, q^{(2)}, w^{(2)}, \mathbf{A}(a_{12}^{(1)}, b_{21}^{(1)})). \quad (4.8)$$

Fix the equation $(1/f^{(2)}, q^{(2)}, w^{(2)})$. If $a_{12}^{(2)} < 1/f_0^{(2)}$, then $a_{12}^{(1)} \leq a_{12}^{(2)} < 1/f_0^{(2)}$. If $f_0^{(1)} > 1/a_{12}^{(1)}$, then $a_{12}^{(2)} \geq a_{12}^{(1)} > 1/f_0^{(2)}$. By Theorem 3.2, one gets that there are exactly N eigenvalues of $(4.2)_2$ – $(4.6)_1$ and $(4.2)_2$ – $(4.6)_2$, and for each $0 \leq n \leq N-1$,

$$\lambda_n(1/f^{(2)}, q^{(2)}, w^{(2)}, \mathbf{A}(a_{12}^{(1)}, b_{21}^{(1)})) \leq \lambda_n(1/f^{(2)}, q^{(2)}, w^{(2)}, \mathbf{A}(a_{12}^{(2)}, b_{21}^{(2)})) = \lambda_n^{(2)}. \quad (4.9)$$

Combining (4.8)–(4.9) yields that (4.7) holds.

With a similar argument to that in the proof of (i) with the assumption (1), one can show that (i) with the assumption (2) and (ii) hold. This completes the proof.

Remark 4.2. One can establish inequalities among eigenvalues of SLPs with different equations and BCs in $\mathcal{O}_{2,4}^{\mathbb{C}}$, $\mathcal{O}_{1,3}^{\mathbb{C}}$, and $\mathcal{O}_{2,3}^{\mathbb{C}}$, separately, with a similar method to that used in Corollary 4.1. We omit their details.

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